CONCORDANCE AND ISOTOPY OF METRICS WITH POSITIVE SCALAR CURVATURE

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Abstract. Two positive scalar curvature metrics g_0, g_1 on a manifold M are psc-isotopic if they are homotopic through metrics of positive scalar curvature. It is well known that if two metrics g_0, g_1 of positive scalar curvature on a closed compact manifold M are psc-isotopic, then they are psc-concordant: i.e., there exists a metric $\bar{g}$ of positive scalar curvature on the cylinder $M \times I$ which extends the metrics g_0 on $M \times \{0\}$ and g_1 on $M \times \{1\}$ and is a product metric near the boundary. The main result of the paper is that if psc-metrics g_0, g_1 on M are psc-concordant, then there exists a diffeomorphism $\Phi : M \times I \to M \times I$ with $\Phi|_{M\times\{0\}} = Id$ (a pseudo-isotopy) such that the metrics g_0 and $(\Phi|_{M\times\{1\}})^* g_1$ are psc-isotopic. In particular, for a simply connected manifold M with $\dim M \geq 5$, psc-metrics g_0, g_1 are psc-isotopic if and only if they are psc-concordant. To prove these results, we employ a combination of relevant methods: surgery tools related to Gromov-Lawson construction, classic results on isotopy and pseudo-isotopy of diffeomorphisms, standard geometric analysis related to the conformal Laplacian, and the Ricci flow.

1. Introduction

1.1. **Motivation.** Let M be a closed manifold. We denote by $\mathcal{R}\mathrm{iem}(M)$ the space of all Riemannian metrics in the C^∞-topology, and by $\mathcal{R}\mathrm{iem}^+(M) \subset \mathcal{R}\mathrm{iem}(M)$ the subspace of metrics g with positive scalar curvature R_g. We use the abbreviation "psc-metric" for "metric with positive scalar curvature".

Throughout the paper, it is assumed that M admits a psc-metric, i.e. $\mathcal{R}\mathrm{iem}^+(M) \neq \emptyset$. The existence of psc-metrics is a well-studied question. In particular, the existence of psc-metrics is well-understood for simply-connected manifolds of dimension at least five, see [16, 29]. It is also well-known that, in general, the space $\mathcal{R}\mathrm{iem}^+(M)$ has many path-components even for simply-connected manifolds, [6, 21], see also [5].

Two psc-metrics $g_0, g_1 \in \mathcal{R}\mathrm{iem}^+(M)$ are *psc-isotopic* if there exists a *smooth* path of psc-metrics $g(t)$, $t \in I = [0,1]$, with $g(0) = g_0$ and $g(1) = g_1$. In that case, we say that the path $g(t)$ is a *psc-isotopy between* g_0 *and* g_1. In fact, psc-metrics g_0 and g_1 are

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psc-isotopic if and only if they belong to the same path-component in $\mathcal{R}\mathrm{iem}^+(M)$ since any continuous path of psc-metrics could be approximated by a smooth one.

Let $\mathrm{Diff}(M)$ be a group of diffeomorphisms of M. The group $\mathrm{Diff}(M)$ acts on the space of metrics $\mathcal{R}\mathrm{iem}(M)$ by pull-back:

$$\mathrm{Diff}(M) \cdot \mathcal{R}\mathrm{iem}(M) \longrightarrow \mathcal{R}\mathrm{iem}(M), \qquad (\varphi, g) \mapsto \varphi^* g.$$

We say that two psc-metrics g_0 and g_1 are *psc-isotopic up to a diffeomorphism* if there exists a diffeomorphism $\varphi \in \mathrm{Diff}(M)$ and a psc-isotopy between g_0 and $\varphi^* g_1$.

Remark. The term "isotopy" has several meanings in smooth topology: there is a standard term *isotopy* for two diffeomorphisms (which is equivalent to the fact that these diffeomorphisms are in the same path-component of $\mathrm{Diff}(M)$). Then there is an *isotopy group* $\mathcal{S}(M \times I)$, which consists of slice-wise diffeomorphisms

$$\Phi : M \times I \to M \times I$$

such that $\Phi|_{M\times\{0\}} = Id_{M\times\{0\}}$. Furthermore, there is a *pseudo-isotopy group*

$$\mathrm{Diff}(M \times I, M \times \{0\})$$

of all diffeomorphisms $\Phi : M \times I \to M \times I$ such that $\Phi|_{M\times\{0\}} = Id_{M\times\{0\}}$, see, say, [18]. Incidentally, all these concepts are relevant to the main subject of this paper. To avoid any confusion, we use the term "psc-isotopy" and its versions for psc-metrics and their equivalence classes up to a diffeomorphism. ◇

Two psc-metrics $g_0, g_1 \in \mathcal{R}\mathrm{iem}^+(M)$ are *psc-concordant* if there exists a psc-metric $\bar{g}$ on $M \times I$ such that

(i) $\bar{g}|_{M\times\{0\}} = g_0$, $\bar{g}|_{M\times\{1\}} = g_1$,
(ii) $\bar{g}$ is a product-metric near the boundary $M \times \{0\} \sqcup M \times \{1\}$.

In that case, we say that the Riemannian manifold $(M \times I, \bar{g})$ is a *psc-concordance* between g_0 and g_1.

Clearly, a psc-isotopy and a psc-concordance are both equivalence relations on the space $\mathcal{R}\mathrm{iem}^+(M)$ of psc-metrics. It is an easy exercise to show that any psc-isotopic metrics are psc-concordant.

From the above definitions, psc-concordance appears to be weaker than psc-isotopy. However, the difference is rather subtle and this is the main subject of this paper. In other words, we study the following

General Question: *Does psc-concordance imply psc-isotopy?*

This question is mentioned as Problem 6.3 in [27], see also [26]. As we shall see in a moment, in general, there is a potential topological obstruction for two psc-concordant metrics to be psc-isotopic: this is closely related to the obstruction which detects a gap between pseudo-isotopy and isotopy of diffeomorphisms. We conjecture that this obstruction should provide many examples of concordant psc-metrics which are not psc-isotopic. On the other hand, we give an affirmative answer to the General Question

modulo of that topological obstruction: two psc-metrics are psc-concordant if and only if they are psc-isotopic up to pseudo-isotopy (see Theorem A). In particular, this implies that the answer to the General Question is alway positive for simply-connected manifolds of dimension at least five (see Theorem B).

There is one more issue that complicates the matter: it turns out that the problem of resolving whether a given psc-concordance could be turned into psc-isotopy is algorithmically unsolvable (see Theorem 1.1).

1.2. **Topological conjecture.** To identify a *potential* topological obstruction, we recall a few definitions and results from smooth topology. Let M be a closed compact manifold without boundary. As above, a diffeomorphism $\Phi : M \times I \to M \times I$ is called a *pseudo-isotopy* if $\Phi|_{M\times\{0\}} = Id_{M\times\{0\}}$. We denote by

$$\mathrm{Diff}(M \times I, M \times \{0\}) \subset \mathrm{Diff}(M \times I)$$

the subgroup of pseudo-isotopies. The group of pseudo-isotopies $\mathrm{Diff}(M \times I, M \times \{0\})$ acts on diffeomorphisms:

$$\mu : \mathrm{Diff}(M \times I, M \times \{0\}) \times \mathrm{Diff}(M) \longrightarrow \mathrm{Diff}(M),$$

where μ sends a pseudo-isotopy $\Phi : M \times I \to M \times I$ and a diffeomorphism $\varphi : M \to M$ to the diffeomorphism

$$\varphi \circ \left(\Phi|_{M\times\{1\}}\right) : M \to M.$$

Then two diffeomorphisms $\varphi_0, \varphi_1 \in \mathrm{Diff}(M)$ are said to be *pseudo-isotopic* if there exists a pseudo-isotopy $\Phi \in \mathrm{Diff}(M \times I, M \times \{0\})$ such that $\mu(\Phi, \varphi_0) = \varphi_1$, i.e.

$$\varphi_0 \circ \left(\Phi|_{M\times\{1\}}\right) = \varphi_1.$$

On the other hand, the group of pseudo-isotopies $\mathrm{Diff}(M \times I, M \times \{0\})$ contains a subgroup $\mathcal{S}(M \times I)$ of *isotopies*, i.e. of diffeomorphisms

$$\Phi \in \mathrm{Diff}(M \times I, M \times \{0\})$$

such that $\pi_I \circ \Phi = \pi_I$, where $\pi_I : M \times I \to I$ is a projection on the second factor. In other words, an isotopy $\Phi \in \mathcal{S}(M \times I)$ is just a smooth path of diffeomorphisms $\Phi_t : M \times \{t\} \to M \times \{t\}$ starting with the identity: $\Phi_0 = Id_{M\times\{0\}}$.

Then two diffeomorphisms $\varphi_0, \varphi_1 \in \mathrm{Diff}(M)$ are said to be *isotopic* if there is an isotopy $\Phi \in \mathcal{S}(M \times I)$ such that $\mu(\Phi, \varphi_0) = \varphi_1$. This is the same as a smooth path $\varphi(t)$ in the group $\mathrm{Diff}(M)$ such that $\varphi(0) = \varphi_0$ and $\varphi(1) = \varphi_1$.

Once we identify $\mathcal{S}(M \times I)$ with the space of smooth paths in $\mathrm{Diff}(M)$ starting at the idenity, we conclude that the isotopy group $\mathcal{S}(M \times I)$ is contractible. Hence the group

$$\pi_0\mathrm{Diff}(M \times I, M \times \{0\})$$

could be identified with the only obstruction to distinguish pseudo-isotopic and isotopic diffeomorphisms. According to J. Cerf [7], the group $\pi_0\mathrm{Diff}(M \times I, M \times \{0\}) = 0$ for simply-connected manifolds M of dimension at least five. However, this group is non-trivial for most other manifolds, and, in general, the group of psuedo-isotopies has highly non-trivial topology.

To see a relationship to psc-metrics, consider a manifold M with

$$\pi_0\mathrm{Diff}(M \times I, M \times \{0\}) \neq 0.$$

Assume that M admits a psc-metric g. We define a psc-metric on the cylinder $\bar{g} = g + dt^2$ on $M \times I$. Then we choose a pseudo-isotopy

$$\Phi : M \times I \to M \times I$$

which represents a nontrivial element in the obstruction group

$$\pi_0\mathrm{Diff}(M \times I, M \times \{0\}).$$

We equip the cylinder $M \times I$ with the psc-metric $\Phi^*\bar{g}$. By construction, the metrics $g_0 = g$ and $g_1 = (\Phi|_{M\times\{1\}})^* g$ are psc-concordant. The question of whether the metrics $g_0 = g$ and g_1 are psc-isotopic or not is open (provided that the diffeomorphism $\Phi|_{M\times\{1\}}$ is not isotopic to the identity).

Topological Conjecture. *Let $\Phi \in \pi_0\mathrm{Diff}(M \times I, M \times \{0\})$ be a nontrivial elemnent such that $\Phi|_{M\times\{1\}}$ is not isotopic to the identity. Then the metrics g_0 and $g_1 = (\Phi|_{M\times\{1\}})^* g$ are not psc-isotopic.*

Remark. Recently W. Steimle has communicated to the author that there exist many nontrivial concordances $\Phi \in \mathrm{Diff}(M \times I, M \times \{0\})$ such that $\Phi|_{M\times\{1\}} = Id_M$, see [33, Theorem 1.2]. Thus the condition that $\Phi|_{M\times\{1\}}$ is not isotopic to the identity is essential in the above conjecture. The author is grateful to W. Steimle for clarifying this issue.

It is worth noting here that the obstruction group $\pi_0\mathrm{Diff}(M \times I, M \times \{0\})$ is often nontrivial; for instance, the obstruction group is "almost always" non-zero if the fundamental group $\pi_1 M$ contains torsion (see, say, [20, 25] for more details). $\diamond$

Remark. In dimension four, D. Ruberman [28] had constructed examples of simply connected manifolds M^4 and psc-concordant psc-metrics g_0 and g_1 which are not psc-isotopic. In that case, the obstruction comes from the Seiberg-Witten invariant and again, it is topological by nature: it detects the gap between an isotopy and a pseudo-isotopy of diffeomorphisms for 4-manifolds. In particular, those examples of psc-metrics are psc-isotopic up to pseudo-isotopy. In particular, the counterexample psc-metrics g_0 and g_1 constructed in [28] both project to the same path-component of the moduli space $\mathcal{R}\mathrm{iem}^+(M)/\mathrm{Diff}(M)$ of psc-metrics (or its version, see [4]).

In other words, the above potential and actual examples of psc-concordant metrics g_0 and g_1 which are not psc-isotopic in the space $\mathcal{R}\mathrm{iem}^+(M)$ are still homotopic in the moduli space of psc-metrics. $\diamond$

1.3. **Algorithmic unsolvability.** There is another important aspect concerning the above General Question. Let $(M \times I, \bar{g})$ be a psc-concordance between psc-metrics g_0 and g_1. If we think about the cylinder $(M \times I, \bar{g})$ isometrically imbedded into Euclidian space, then it might be extremely long and could contain very complicated features which cannot be effectively described analytically or topologically. In dealing with these issues, it is important to keep in mind the following result:

Theorem 1.1. (M. Gromov) *A problem of deciding whether two psc-concordant psc-metrics are psc-isotopic is algorithmically unsolvable.*

Remark. The proof of Theorem 1.1 is based on a well-known fact, namely, that the problem of recognizing a trivial group out of given finite sets of generators and relations is algorithmically unsolvable. To get to a psc-concordance, we take finite "unrecognizable" sets of generators and relations; this gives us finite 2-complex K, which has a unique zero cell, as many 1-cells as the number of generators, and with 2-cells attached according to the relations. By construction, $\pi_1 K = 0$. We embed K into the Euclidean space $\mathbf{R}^5$ and denote by $T(K)$ its closed tubular neighbourhood in $\mathbf{R}^5$. Then we double $T(K)$ to form a closed, simply connected compact manifold $X^5 = T(K) \cup_{\partial T(K)} -T(K)$. The product $X^5 \times S^3$ has an obvious psc-metric. By construction, the manifold $X^5 \times S^3$ is simply-connected, and there is a surgery (of an appropriate codimension) to turn $X^5 \times S^3$ into a homotopy sphere Σ^8 equipped with a psc-metric. Then, after deleting two small disks, one constructs an *exotic* psc-concordance $(\tilde{\Sigma}^8, \tilde{g})$ between two round standard spheres (S^7, h_0), see Fig. 1. It is indeed exotic since there is no algorithm which would turn that psc-concordance into psc-isotopy: otherwise, it would recognize along the way that the original system of generators and relations determines a trivial group. $\diamond$

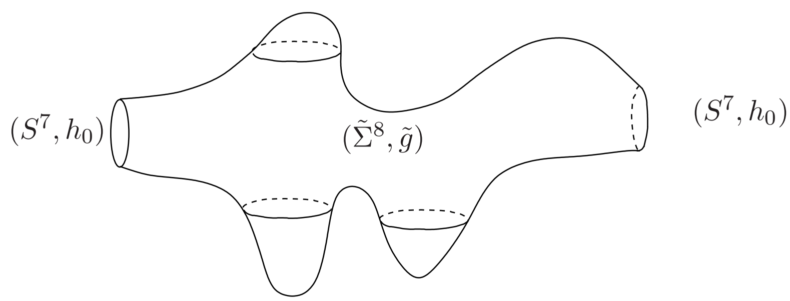

Figure 1. An exotic psc-concordance

In particular, Theorem 1.1 implies that in order to make any progress on whether a psc-concordance implies a psc-isotopy, we have to employ some tools which are "non-algorithmic" by their nature, such as surgery.

1.4. **Main results.** Since in general, there are topological obstructions to find a psc-isotopy for psc-concordant metrics, we would like to separate the *geometric* issues from the *topological* ones concerning the problem of whether psc-concordance implies psc-isotopy. Here is the first main result:

Theorem A. *Let M be a closed compact manifold with $\dim M \geq 3$. Then, for any two psc-concordant metrics $g_0, g_1 \in \mathcal{R}\mathrm{iem}^+(M)$ there exists a pseudo-isotopy*

$$\Phi \in \mathrm{Diff}(M \times I, M \times \{0\})$$

such that the psc-metrics g_0 and $\Phi^* g_1 = (\Phi|_{M \times \{1\}})^* g_1$ are psc-isotopic.

According to J. Cerf's result [7], $\pi_0 \mathrm{Diff}(M \times I, M \times \{0\}) = 0$ for simply connected manifold M with $\dim M \geq 5$. Hence, in that case, there is no obstruction for two pseudo-isotopic diffeomorphisms to be isotopic. This gives the second main result as a corollary of Theorem A.

Theorem B. *Let M be a closed simply connected manifold with* $\dim M \geq 5$*. Then two psc-metrics g_0 and g_1 on M are psc-isotopic if and only if the metrics g_0, g_1 are psc-concordant.*

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2. The strategy to prove Theorem A

2.1. **First steps.** First, we would like to specify the statement of Theorem A for a given manifold M. We use the abbreviation "$(\mathbf{C}\Longleftrightarrow\mathbf{I})(M)$" for the following statement:

"Let $g_0, g_1 \in \mathcal{R}\mathrm{iem}^+(M)$ be any psc-concordant metrics. Then, there exists a pseudo-isotopy $\Phi \in \mathrm{Diff}(M\times I, M\times\{0\})$ such that the psc-metrics g_0 and $\Phi^* g_1 = (\Phi^*\bar{g})|_{M\times\{1\}}$ are psc-isotopic."

It turns out that it is much easier to prove the statement $(\mathbf{C}\Longleftrightarrow\mathbf{I})(M)$ if the manifold M does not admit any Ricci-flat metric. To reduce Theorem A to such a case, we have to make two more steps as follows.

2.2. **PSC-concordance-isotopy surgery Theorem.** Let M be a closed manifold, $\dim M = n-1$, and $S^p \subset M$ be a embedded sphere in M with trivial normal bundle. We assume that it's embedded together with it's tubular neighbourhood $S^p\times D^{q+1} \subset M$. Here $p+q+1 = n-1$. Then we denote by M' the manifold which is resulting from the surgery along the sphere S^p:

$$M' = (M \setminus (S^p \times D^{q+1})) \cup_{S^p\times S^q} (D^{p+1} \times S^q).$$

The codimension of the sphere $S^p \subset M$ is called a *codimension of the surgery*. In the above terms, the codimension of the above surgery is $(q+1)$.

Example. Let $M = S^k$, $k \geq 4$. Then, it is easy to make a surgery of codimension k on S^k to construct $M' = S^1 \times S^{k-1}$, see Fig. 2. On the other hand, there is a second surgery which recovers S^k from $S^1 \times S^{k-1}$, see Fig. 3. We notice that both surgeries are of codimension at least three, provided $k \geq 4$.

There is more general construction: a surgery along a submanifold $\Sigma \subset M$ which is embedded into M together with a trivial normal bundle, i.e. $\Sigma \times D^{q+1} \subset M$. We

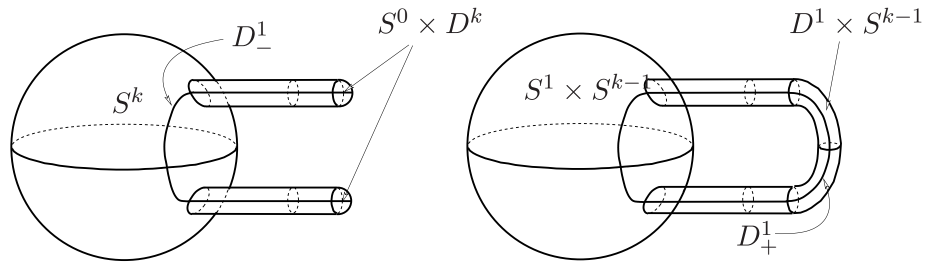

FIGURE 2. The surgery $S^k \Longrightarrow S^1 \times S^{k-1}$.

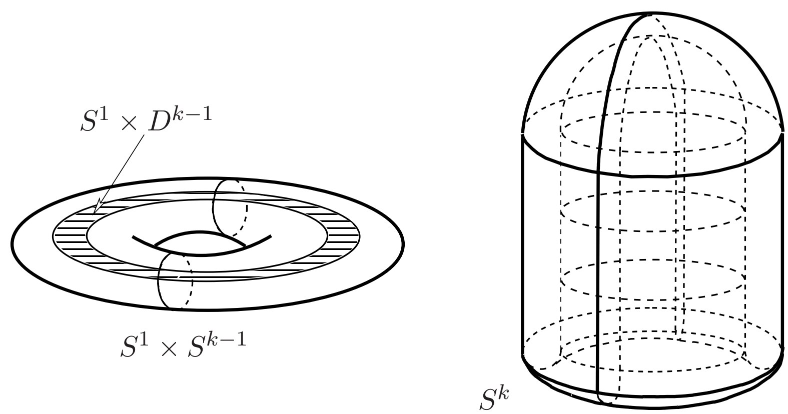

FIGURE 3. The surgery $S^1 \times S^{k-1} \Longrightarrow S^k$.

assume that $\Sigma = \partial X$. Then we form a new manifold:

$$M'_{\Sigma,X} = \big(M \setminus (\Sigma \times D^{q+1})\big) \cup_{\Sigma \times S^q} (X \times S^q).$$

Then we say that the manifold $M'_{\Sigma,X}$ is constructed constructed out of M by a surgery of codimension $q+1$.

Example. For instance, there is a surgery

$$S^1 \times S^{k-1} \Longrightarrow S^1 \times S^1 \times S^{k-2},$$

along $\Sigma = S^1 \times S^0$, where $\Sigma = \partial(S^1 \times (D^1 \times S^{k-2})$, such that:

$$\begin{aligned} S^1 \times S^1 \times S^{k-2} &= \big((S^1 \times S^{k-1}) \setminus (S^1 \times S^0 \times D^{k-1})\big) \cup_{S^1 \times S^0 \times S^{k-2}} (S^1 \times (D^1 \times S^{k-2})) \\ &= S^1 \times \big((S^{k-1} \setminus (S^0 \times D^{k-1})) \cup_{S^0 \times S^{k-2}} (D^1 \times S^{k-2})\big). \end{aligned}$$

Similarly, there is a surgery

$$S^1 \times S^1 \times S^{k-2} \Longrightarrow S^1 \times S^{k-1}$$

along $\Sigma' = S^1 \times S^1$ with $X = S^1 \times D^2$.

Definition 2.1. In the case if $\Sigma = S^p$ and $X = D^{p+1}$ or, respectively, $\Sigma = S^k \times S^{p-k}$ and $X = S^k \times D^{p-k+1}$, we say that the surgery along Σ is *spherical* or, respectively, *almost spherical*.

Definition 2.2. Let M and M' be manifolds such that:

- M' can be constructed out of M by a finite sequence of spherical or almost spherical surgeries of codimension at least three, and
- M can be constructed out of M' by a finite sequence of spherical or almost spherical surgeries of codimension at least three.

Then, we say that M *and* M' *are related by admissible surgeries.*

Remark. In particular, the manifolds S^k and $T^{k-3} \times S^3$ are related by admissible surgeries if $k \geq 4$. Moreover, the manifolds

$$M \cong M\#S^k \quad \text{and} \quad M' = M\#(T^{k-3} \times S^3)$$

are also related by admissible surgeries. $\diamond$

We prove the following result in Section 9:

Theorem 2.3. *Let M and M' be closed manifold which are related by admissible surgeries. Then the statements $(\mathbf{C}{\Longleftrightarrow}\mathbf{I})(M)$ and $(\mathbf{C}{\Longleftrightarrow}\mathbf{I})(M')$ are equivalent.*

In particular, Theorem 2.3 implies the following result:

Corollary 2.4. *Let M be a closed manifold with $\dim M = k \geq 4$. We let $M' := M\#(T^{k-3} \times S^3)$. Then the statements $(\mathbf{C}{\Longleftrightarrow}\mathbf{I})(M)$ and $(\mathbf{C}{\Longleftrightarrow}\mathbf{I})(M')$ are equivalent.*

2.3. **Surgery and Ricci-flatness.** As it turns out, it is easy to use surgery in order to construct a manifold which does not admit any Ricci-flat metric. The following result follows directly from [8, Theorem 3]:

Theorem 2.5. *Let M be a closed connected manifold with $\dim M = k \geq 4$. Then the manifold*

$$M' = M\#(S^3 \times T^{k-3})$$

does not admit a Ricci-flat metric.

Corollary 2.4 and Theorem 2.5 imply that it is enough to prove Theorem A under the restriction that a manifold M does not admit a Ricci-flat metric.

2.4. **The simplest case when psc-concordance implies psc-isotopy.** Now we may return to Theorem A. We start with a psc-concordance $(M \times I, \bar{g})$. Let

$$\pi_M : M \times I \to M, \quad \pi_I : M \times I \to I \tag{1}$$

be projections on the first and the second factors. This gives us a coordinate system (x, t) on the product $M \times I$.

We assume that the metric $\bar{g}$ is given as $\bar{g} = g_t + dt^2$ with respect to this coordinate system. Here $g_t = \bar{g}|_{M\times\{t\}}$. Moreover, we assume that the mean curvature H_{g_t} along the hypersurface $M \times \{t\}$ is identically zero for each $t \in I$.

As it turns out, this is an ideal situation which guarantees that the metrics g_t have positive scalar curvature for all $t \in I$. Indeed, if $\bar{g} = g_t + dt^2$, then the Gauss formula could be written as follows:

$$R_{\bar{g}} = R_{g_t} + 2\partial_0 H_t - H_t^2 - |A_t|^2, \tag{2}$$

where A_t is the second fundamental form of the hypersurface $M \times \{t\}$, H_t is the mean curvature along $M \times \{t\}$, and $\partial_0 H_t$ its derivative in the t-direction. Thus if the hypersurfaces $M \times \{t\}$ are minimal for all $t \in I$ (i.e. $H_t \equiv 0$), then (2) implies

$$R_{\bar{g}} = R_{g_t} - |A_t|^2.$$

Hence $R_{g_t} > 0$ for all $t \in I$ if $R_{\bar{g}} > 0$. We summarize these observations:

Proposition 2.6. *Let $(M \times I, \bar{g})$ be a Riemannian manifold such that*

- **(a)** $R_{\bar{g}} > 0$*;*
- **(b)** $\bar{g} = g_t + dt^2$ *with respect to the coordinate system given by* (1)*;*
- **(c)** $H_{g_t} \equiv 0$ *for all* $t \in I$*.*

Then the metrics g_t have positive scalar curvature for all $t \in I$. In particular, the family of psc-metrics $\{g_t\}$ provides psc-isotopy between g_0 and g_1.

Clearly the conditions **(a)**, **(b)** and **(c)** are too strong to expect that for given psc-metrics g_0, g_1 on M, one can easily find a psc-concordance $(M \times I, \bar{g})$ like that. Moreover, it is very difficult to balance the conditions **(a)**, **(b)** and **(c)**. For example, a small conformal change of the metric $\bar{g}$ maintains positivity of the scalar curvature, but it easily violates both of the conditions **(b)** and **(c)**.

2.5. **Slicing functions.** Now we are getting close to a central problem here: for a given psc-concordance $(M \times I, \bar{g})$ between psc-metrics g_0 and g_1, we should look for a *slicing function* $\bar{\alpha} : M \times I \to I$ such that the curve of Riemannian manifolds (M_t, g_t) provides a desired psc-isotopy. Here $M_t = \bar{\alpha}^{-1}(t)$, and $g_t = \bar{g}|_{M_t}$.

Let M be a closed smooth manifold. We consider the direct product $M \times I$ and the projection $\pi_I : M \times I \to I$ on the second factor.

Definition 2.7. A slicing function $\bar{\alpha} : M \times I \to I$ is a smooth function such that

- **(i)** it has no critical points;
- **(ii)** it agrees with the projection $\pi_I : M \times I \to I$ near the boundary
$$\partial(M \times I) = M \times \{0\} \sqcup M \times \{1\},$$
in particular, $\bar{\alpha}^{-1}(0) = M \times \{0\}$ and $\bar{\alpha}^{-1}(1) = M \times \{1\}$.

We denote by $\mathcal{E}(M \times I)$ the space of slicing functions with the Whitney topology. We will review necessary relults on the space of slicing functions in the next section.

2.6. **Sufficient conditions.** Let $\dim M = n-1 \geq 4$. Let $\bar{C}$ be a conformal class of metrics on $M \times I$, and $C_0 = \bar{C}|_{M\times\{0\}}$ and $C_1 = \bar{C}|_{M\times\{1\}}$. We say that a conformal class C on M is *positive* if it contains a psc-metric. Then we say that $(M \times I, \bar{C})$ is a *conformal psc-concordance between positive conformal classes C_0 and C_1* if there exists a psc-metric $\bar{g} \in \bar{C}$ with zero mean curvature along the boundary. As it turns out, conformal psc-concordance is equivalent to psc-concordance (see Theorem 3.2 and Section 3 for more details).

Given a conformal psc-concordance $(M\times I, \bar{C})$, we choose a metric $\bar{g} \in \bar{C}$ with minimal boundary condition, which does not necesarily have positive scalar curvature. Next, we choose a slicing function $\bar{\alpha} : M \times I \to I$, $\bar{\alpha} \in \mathcal{E}(M \times I)$. In particular, the slicing function τ gives the coordinates (x,t) on $M \times I$. Then for each $t < s$, we define a manifold $W^*_{t,s} = \bar{\alpha}^{-1}([t,s])$ equipped with a metric $\bar{g}^*_{t,s} = \bar{g}|_{W_{t,s}}$.

Furthemore, for each t,s, $0 \leq t < s \leq 1$, we let $\xi_{t,s} : [0,1] \to [t,s]$ be a linear function sending $\tau \mapsto (1-\tau)t + \tau s$. This gives a diffeomorphism

$$\bar{\xi}_{t,s} : M \times [0,1] \to M \times [s,t], \qquad (x,\tau) \mapsto (x, \xi_{t,s}(\tau)).$$

We use the map $\bar{\xi}_{t,s}$ to stretch the manifold $(W^*_{t,s}, \bar{g}^*_{t,s})$ in the horizontal direction, and denote by $(W_{t,s}, \bar{g}_{t,s})$ the resulting stretched manifold, $W_{t,s} = M \times I$, see Fig. 4. Then $\partial W_{t,s} = M_t \sqcup M_s$, where $M_\tau = \bar{\alpha}^{-1}(\tau)$, $\tau = t,s$. Let $A_{\bar{g}_{t,s}}$ be the second

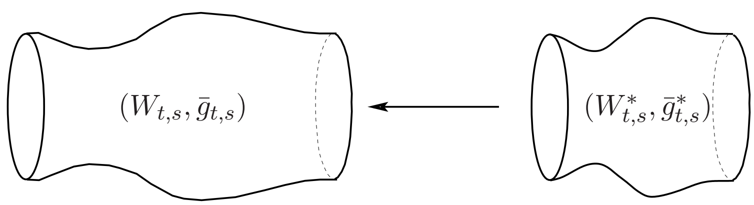

FIGURE 4. Stretching $(W^*_{t,s}, \bar{g}^*_{t,s})$ to get $(W_{t,s}, \bar{g}_{t,s})$.

fundamental form of the metric $\bar{g}_{t,s}$ along the boundary. We denote by $h_{\bar{g}_{t,s}} = \frac{1}{n-1} \operatorname{tr} A_{\bar{g}_{t,s}}$ the normalized mean curvature along the boundary $\partial W_{t,s} = M_t \sqcup M_s$. Let

$$L_{\bar{g}_{t,s}} = a_n \Delta_{\bar{g}_{t,s}} + R_{\bar{g}_{t,s}}, \quad \text{where} \quad a_n = \tfrac{4(n-1)}{n-2},$$

be the conformal Laplacian on the manifold $(W_{t,s}, \bar{g}_{t,s})$.

We denote by $\lambda_1(L_{\bar{g}_{t,s}})$ the principal eigenvalue of the minimal boundary problem:

$$\left\{\begin{array}{lcll} L_{\bar{g}_{t,s}} u & = & a_n \Delta_{\bar{g}_{t,s}} u + R_{\bar{g}_{t,s}} u = \lambda_1(L_{\bar{g}_{t,s}}) u & \text{on } W_{t,s} \\ B_{\bar{g}_{t,s}} u & = & \partial_\nu u + \frac{n-2}{2} h_{\bar{g}_{t,s}} u = 0 & \text{on } \partial W_{t,s}. \end{array}\right. \tag{3}$$

Here ∂_ν is the outward unit normal vector field along the boundary. We obtain a continious function $\Lambda_{(M\times I, \bar{g}, \bar{\alpha})}(t,s) = \lambda_1(L_{\bar{g}_{t,s}})$ (see Section 3.8 for details).

Now the idea is to replace the sufficient conditons **(a)**, **(b)** and **(c)** from Proposition 2.6 with the non-negativity of the function $\Lambda_{(M\times I, \bar{g}, \bar{\alpha})}$. It turns out this is enough provided the manifold M does not admit a Ricci-flat metric.

Theorem 2.8. *Let M be a closed manifold with $\dim M = n-1 \geq 4$ which does not admit a Ricci-flat metric. Assume that $(M \times I, \bar{C})$ is a conformal psc-concordance between positive conformal classes C_0 and C_1, $\bar{g} \in \bar{C}$ is a metric with zero mean curvature along the boundary, and $\bar{\alpha} : M \times I \to I$ is a slicing function such that $\Lambda_{(M\times I, \bar{g}, \bar{\alpha})} \geq 0$. Then any two psc-metrics $g_0 \in C_0$ and $g_1 \in C_1$ are psc-isotopic up to pseudo-isotopy, i.e., there exists a pseudoisotopy*

$$\Phi : M \times I \to M \times I$$

such that the psc-metrics g_0 and $(\Phi|_{M\times\{1\}})^* g_1$ are psc-isotopic.

2.7. **Comments on Theorem 2.8.** We would like to answer the following questions:

- **(1)** Why do we need the condition that M does not admit a Ricci-flat metric?
- **(2)** How does a pseudo-isotopy appear here?

(1) Assume the slicing function $\bar{\alpha}$ coincides with the projection $\pi_I : M \times I \to I$. Furthemore, we assume that the metric $\bar{g}$ is given as $\bar{g} = g_\tau + d\tau^2$. Consider now the conformal Laplacian $L_{\bar{g}_{t,s}}$ on $W_{t,s}$ with the minimal boundary condition. Assuming that $\Lambda(t,s) = \lambda_1(L_{\bar{g}_{t,s}}) \geq 0$ for all pairs $t < s$, one can show that the conformal Laplacian L_{g_t} on the slice (M_t, g_t) has nonnegative principal eigenvalue $\lambda_1(L_{g_t}) \geq 0$ for each t.

Then we find positive eigenfunctions $u(t)$ corresponding to the eigenvalue $\lambda_1(L_{g_t})$; the functions $u(t)$ depend continuously on t. We make a slice-wise conformal deformation $\hat{g}_t = u(t)^{\frac{4}{n-3}} g_t$, then

$$R_{\hat{g}_t} = u(t)^{-\frac{4}{n-3}} \lambda_1(L_{g_t}) = \begin{cases} > 0 & \text{if } \lambda_1(L_{g_t}) > 0, \\ \equiv 0 & \text{if } \lambda_1(L_{g_t}) = 0. \end{cases}$$

If $\lambda_1(L_{g_t}) > 0$ for all t, then we have obtained a path of psc-metrics.

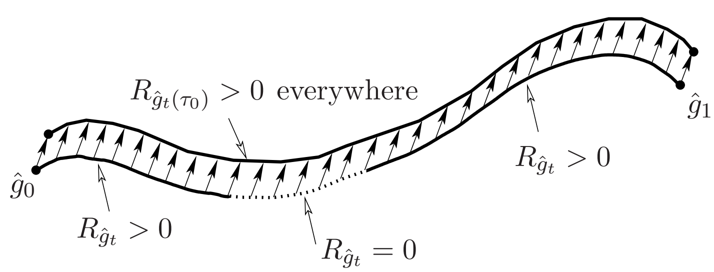

FIGURE 5. Ricci flow applied to the path $\hat{g}_t$.

If $\lambda_1(L_{g_t}) = 0$, we require the condition that M does not admit a Ricci flat metric. In that case, if the metric $\hat{g}_t$ is scalar flat, it cannot be Ricci-flat. Thus for each t we can start the Ricci flow $\hat{g}_t(\tau)$ with the initial metric $\hat{g}_t(0) = \hat{g}_t$, so that short-time existence of the Ricci Flow yields a path of psc-metrics $\hat{g}_t(\tau_0)$, where the parameter τ_0 is small (see Fig. 5). ◇

(2) Now we let $\bar{\alpha}$ be an arbitrary slicing function, but we assume that the metric $\bar{g}$ and the function $\bar{\alpha}$ are coupled as follows. First, we assume that $|\nabla \bar{\alpha}|_{\bar{g}} = 1$. We consider a

trajectory γ_x of the gradient vector field $\nabla\bar{\alpha}$ satisfying the initial condition $\gamma_x(0)=x$, $x\in M\times\{0\}$. This generates a pseudo-isotopy $\Phi : M\times I\to M\times I$ given by the formula

$$\Phi : (x,t)\mapsto (\pi_M(\gamma_x(t)),\pi_I(\gamma_x(t)) := (y,s).$$

Then we obtain a metric $\tilde{g} = (\Phi^{-1})^*(\bar{g}) = g_s + ds^2$. Thus, the condition $|\nabla\bar{\alpha}|_{\bar{g}} = 1$ converts to the condition **(c)**. Then one can generalize the above argument we use in **(1)** to show that g_0 and g_1 are isotopic up to pseudo-isotopy. ◇

2.8. **Necessary condition.** Here is the necessary condition:

Theorem 2.9. *Let M be a closed manifold with $\dim M = n-1\geq 3$, and $C_0, C_1\in\mathcal{C}(M)$ be conformaly psc-concordant conformal classes. Then there exist*

- **(i)** *a conformal psc-concordance $(M\times I,\bar{C})$ between C_0 and C_1,*
- **(ii)** *a metric $\bar{g}\in\bar{C}$ with minimal boundary condition,*
- **(iii)** *a slicing function $\bar{\alpha}\in\mathcal{E}(M\times I)$*

such that $\Lambda_{(M\times I,\bar{g},\bar{\alpha})}\geq 0$.

Remark. It is easy to see that Theorem 2.9 together with Theorem 2.8 (taking into account Corollary 2.4 and Theorem 2.5) imply Theorem A. ◇

We dedicate Sections 4, 5 and 6 to prepare for the proof of Theorem 2.9.

3. Geometrical and topological preliminaries

3.1. **Conformal psc-concordance.** Let M be a closed smooth manifold with $\dim M = n-1\geq 3$. Let $\mathcal{C}(M)$ be the space of conformal classes of Riemannian metrics on M. We denote by $\pi : \mathcal{R}\mathrm{iem}(M)\to\mathcal{C}(M)$ the canonical projection map which sends a metric g to its conformal class $[g]$. Recall that a conformal class $C\in\mathcal{C}(M)$ is called *positive* if there exists a psc-metric $g\in C$. This is equivalent to positivity of the Yamabe constant $Y_C(M)$ which is defined by the formula:

$$Y_C(M) = \inf_{g\in C}\frac{\int_M R_g d\sigma_g}{\mathrm{Vol}_g(M)^{\frac{n-3}{n-1}}}.$$

We denote the space of all positive conformal classes by $\mathcal{C}^+(M)$. It is known (see, say, [1, Theorem 7.1]) that the projection $\pi : \mathcal{R}\mathrm{iem}(M)\to\mathcal{C}(M)$ induces weak homotopy equivalence:

$$\mathcal{R}\mathrm{iem}^+(M)\simeq\mathcal{C}^+(M).$$

In particular, the spaces $\mathcal{R}\mathrm{iem}^+(M)$ and $\mathcal{C}^+(M)$ have the same number of path components.

Now let $\bar{C}$ be a conformal class on the cylinder $M\times I$. We denote:

$$C_0 = \bar{C}|_{M\times\{0\}},\qquad C_1 = \bar{C}|_{M\times\{1\}}.$$

For a given conformal class $\bar{C}$, we denote by $\bar{C}^0$ the subclass

$$\bar{C}^0 = \{\ \bar{g}\in\bar{C}\mid H_{\bar{g}}\equiv 0\ \text{ along the boundary }\}\subset\bar{C}.$$

It is easy to see that the subclass $\bar{C}^0$ is always non-empty, see [9]. Here $H_{\bar{g}}$ is the mean curvature function. In this setting, a *relative Yamabe constant*

$$Y_{\bar{C}}(M\times I, M\times\{0\}\sqcup M\times\{1\}; C_0\sqcup C_1) = \inf_{\bar{g}\in\bar{C}^0} \frac{\int_{M\times I} R_{\bar{g}} d\sigma_{\bar{g}}}{\mathrm{Vol}_{\bar{g}}(M\times I)^{\frac{n-2}{n}}}$$

is well-defined, see [9].

Definition 3.1. Let $C_0, C_1 \in \mathcal{C}^+(M)$ be two positive conformal classes. We say that C_0 and C_1 are *conformally psc-concordant* if there exists a conformal class $\bar{C}$ on $M\times I$ with $C_0 = \bar{C}|_{M\times\{0\}}$, $C_1 = \bar{C}|_{M\times\{1\}}$ such that

$$Y_{\bar{C}}(M\times I, M\times\{0\}\sqcup M\times\{1\}; C_0\sqcup C_1) > 0.$$

In that case, a conformal manifold $(M\times I, \bar{C})$ is called a *conformal psc-concordance* between the classes C_0, C_1.

It turns out the concepts of psc-concordance between psc-metrics and positive conformal classes are equivalent:

Theorem 3.2. (See [2, Corollary D]) *Let M be a closed manifold with $\dim M\geq 2$, and g_0, g_1 be two psc-metrics. Then the psc-metrics g_0, g_1 are psc-concordant if and only if the conformal classes $C_0=[g_0]$ and $C_1=[g_1]$ are conformally psc-concordant.*

3.2. **The space of non-negative conformal classes.** Let M be as above, a closed manifold with $\dim M = n-1\geq 3$. We denote by $\mathcal{C}^{\geq 0}(M)$ the space of non-negative conformal classes:

$$\mathcal{C}^{\geq 0}(M) = \{\ C\in\mathcal{C}(M)\ |\ Y_C(M)\geq 0\ \}.$$

There is a natural embedding $i:\mathcal{C}^+(M)\hookrightarrow\mathcal{C}^{\geq 0}(M)$.

Lemma 3.3. *Assume a closed manifold M does not admit a Ricci-flat metric. Then the embedding $i:\mathcal{C}^+(M)\hookrightarrow\mathcal{C}^{\geq 0}(M)$ induces an isomorphism $i_*:\pi_0\mathcal{C}^+(M)\overset{\cong}{\hookrightarrow}\pi_0\mathcal{C}^{\geq 0}(M)$.*

Proof. If $C_0, C_1\in\mathcal{C}^+(M)$ are in the same path-component of $\mathcal{C}^+(M)$, then obviously C_0, C_1 are also in the same path-component of $\mathcal{C}^{\geq 0}(M)$.

Assume that $C_0, C_1\in\mathcal{C}^+(M)$ are in the same path-component of $\mathcal{C}^{\geq 0}(M)$, and C_t, $0\leq t\leq 1$, is a continuous path in $\mathcal{C}^{\geq 0}(M)$ between C_0 and C_1. We choose a continuous path of metrics g_t with $g_t\in C_t$ lifting the path C_t. We may assume that $\mathrm{Vol}_{g_t}(M)=1$. Since $\lambda_1(L_{g_t})\geq 0$, we find a family of eigenfunctions u_t such that

$$L_{g_t}u_t = \lambda_1(L_{g_t})u_t, \qquad \int_M u_t^2 d\sigma_{g_t} = 1.$$

Then the family of metrics $\tilde{g}_t = u_t^{\frac{4}{n-3}} g_t$ provides a different lift of the path C_t, and

$$R_{\tilde{g}_t} = \begin{cases} >0 & \text{if } \lambda_1(L_{g_t})>0,\\ =0 & \text{if } \lambda_1(L_{g_t})=0.\end{cases}$$

We start a family of Ricci flows $\tilde{g}_t(\tau)$ with the initial values $\tilde{g}_t(0)=\tilde{g}_t$. Since M does not admit a Ricci-flat metric, there is a short-time solution $\tilde{g}_t(\tau)$ continuously depending on

the initial values. This gives a path of psc-metrics connecting $\tilde{g}_0$ and $\tilde{g}_1$ and consequently, a path of positive conformal classes between C_0 and C_1. $\square$

We do not need the following result here. However, it has an independent interest.

Theorem 3.4. *Let M be a closed manifold with $\dim M \geq 3$ which does not admit a Ricci-flat metric. Then the embedding $i : \mathcal{C}^+(M) \hookrightarrow \mathcal{C}^{\geq 0}(M)$ induces a homotopy equivalence.*

Remark. The proof of Theorem 3.4 is essentially the same as of Lemma 3.3: instead of a path of conformal classes one should consider a compact family $\{C_\zeta\}_{\zeta\in Z}$ of conformal classes, $C_\zeta \in \mathcal{C}^{\geq 0}(M)$. Then again a family of short-time solutions of the corresponding Ricci flows provides a deformation of the family $\{C_\zeta\}_{\zeta\in Z}$ into the space $\mathcal{C}^+(M)$. This gives weak homotopy equivalence, and according to [24], this also implies an actual homotopy equivalence between the spaces $\mathcal{C}^+(M)$ and $\mathcal{C}^{\geq 0}(M)$. $\diamond$

3.3. **Conformal Laplacian and minimal boundary condition.** Here we recall necessary definitions on the conformal Laplacian on a manifold with boundary.

Let $(W,\bar{g})$ be a manifold with boundary ∂W, $\dim W = n$. We denote by $A_{\bar{g}}$ the second fundamental form along ∂W, by $H_{\bar{g}} = \operatorname{tr} A_{\bar{g}}$ the mean curvature along ∂W, and by $h_{\bar{g}} = \frac{1}{n-1}H_{\bar{g}}$ the "normalized" mean curvature. Also we denote by ∂_ν the directional derivative with respect to the outward unit normal vector field along the boundary ∂W.

Let $\tilde{g} = u^{\frac{4}{n-2}}\bar{g}$ be a conformal metric. Then we have the following standard formulas for the scalar and mean curvatures:

$$
\begin{array}{rcl}
R_{\tilde{g}} &=& u^{-\frac{n+2}{n-2}}\left(a_n\Delta_{\bar{g}}u + R_{\bar{g}}u\right) = u^{-\frac{n+2}{n-2}}L_{\bar{g}}u, \qquad a_n = \frac{4(n-1)}{n-2} \\
h_{\tilde{g}} &=& \frac{2}{n-2}u^{-\frac{n}{n-2}}\left(\partial_\nu u + \frac{n-2}{2}h_{\bar{g}}u\right) = u^{-\frac{n}{n-2}}B_{\bar{g}}u.
\end{array} \tag{4}
$$

Then the *minimal boundary problem* on $(W,\bar{g})$ is given as

$$
\left\{
\begin{array}{rcll}
L_{\bar{g}}u &=& a_n\Delta_{\bar{g}}u + R_{\bar{g}}u = \lambda_1 u & \text{on } W, \\
B_{\bar{g}}u &=& \partial_\nu u + \frac{n-2}{2}h_{\bar{g}}u = 0 & \text{on } \partial W,
\end{array}
\right. \tag{5}
$$

where λ_1 is the corresponding principal eigenvalue. If u is a smooth and positive eigenfunction corresponding to the principal eigenvalue λ_1, i.e. $L_{\bar{g}}u = \lambda_1 u$, and $\tilde{g} = u^{\frac{4}{n-2}}\bar{g}$, then

$$
\left\{
\begin{array}{rcll}
R_{\tilde{g}} &=& u^{-\frac{n+2}{n-2}}L_{\bar{g}}u = \lambda_1 u^{-\frac{4}{n-2}} & \text{on } W \\
h_{\tilde{g}} &=& u^{-\frac{n}{n-2}}B_{\bar{g}}u = 0 & \text{on } \partial W.
\end{array}
\right. \tag{6}
$$

3.4. **Slicing functions and pseudoisotopies.** Let M be a closed smooth manifold, as above. We take a direct product $M\times I$ and denote by $\pi_I : M\times I \to M$ the projection on the second factor.

According to Definition 2.7, a slicing function $\bar{\alpha} : M \times I \to I$ is a smooth function such that it has no critical points and it agrees with the projection $\pi_I : M \times I \to I$ near the boundary $\partial(M \times I) = M \times \{0\} \sqcup M \times \{1\}$.

We denote by $\mathcal{E}(M \times I)$ the space of slicing functions in the *Whitney topology* (known also as *weak C^∞-topology*, see [18, Chapter 1]). We denote by $\mathrm{Diff}(M \times I)$ the group of diffeomorphisms of $M \times I$ endowed also with the Whitney topology.

Then we denote by

$$\mathrm{Diff}(M \times I, M \times \{0\}) \subset \mathrm{Diff}(M \times I)$$

the subgroup of diffeomorphisms

$$\bar{\varphi} : M \times I \longrightarrow M \times I$$

such that $\bar{\varphi}|_{M \times \{0\}} = Id_{M \times \{0\}}$.

The group $\mathrm{Diff}(M \times I, M \times \{0\})$ is known as group of *pseudo-isotopies*. There is a natural map

$$\sigma : \mathrm{Diff}(M \times I, M \times \{0\}) \longrightarrow \mathcal{E}(M \times I) \tag{7}$$

which sends a diffeomorphism $\bar{\varphi} : M \times I \longrightarrow M \times I$ to the function

$$\sigma(\bar{\varphi}) = \pi_I \circ \bar{\varphi} : M \times I \xrightarrow{\bar{\varphi}} M \times I \xrightarrow{\pi_I} I \ ,$$

where $\pi_I : M \times I \to I$ is as above, the projection on the second factor.

Theorem 3.5. (J. Cerf, [7]) *The map*

$$\sigma : \mathrm{Diff}(M \times I, M \times \{0\}) \xrightarrow{\simeq} \mathcal{E}(M \times I)$$

is fibration and induces homotopy equivalence.

Remark. It is easy to see that σ is homotopy equivalence. Indeed, consider the fiber over the function $\pi_I : M \times I \to I$:

$$\sigma^{-1}(\pi_I) = \{ \ \bar{\varphi} \in \mathrm{Diff}(M \times I, M \times \{0\}) \mid \pi_I \circ \bar{\varphi} = \pi_I \ \}.$$

Then the space $\sigma^{-1}(\pi_I)$ is homeomorphic to the following space of paths:

$$\{ \ \gamma : I \to \mathrm{Diff}(M) \mid \gamma(0) = Id_M \ \}.$$

This homeomorphism is given by a map which sends a diffeomorphism $\bar{\varphi} \in \sigma^{-1}(\pi_I)$ to the path $\gamma_t : M \to M$, where $\gamma_t(x) = \bar{\varphi}(t, x)$. By definition, $\gamma_0 = Id_M$. Thus the space $\sigma^{-1}(\pi_I)$ is contractible. $\diamond$

We denote by $\mathcal{F}(M \times I)$ the space of all smooth functions $M \times I \to I$ which agree with the projection $\pi_I : M \times I \to I$ near the boundary $\partial(M \times I) = M \times \{0\} \sqcup M \times \{1\}$.

Clearly the space $\mathcal{F}(M \times I)$ is convex, and $\mathcal{E}(M \times I) \subset \mathcal{F}(M \times I)$. Thus, we have the isomorphism:

$$\pi_q(\mathcal{E}(M \times I)) \cong \pi_{q+1}(\mathcal{F}(M \times I), \mathcal{E}(M \times I)) \ . \tag{8}$$

The isomorphism

$$\pi_0(\mathcal{E}(M \times I)) \cong \pi_1(\mathcal{F}(M \times I), \mathcal{E}(M \times I))$$

is relevant to our story and has the following geometric interpretation.

Two diffeomorphisms $\varphi_0, \varphi_1 \in \mathrm{Diff}(M)$ are said to be *isotopic* if there is a smooth path $\varphi(t)$ in $\mathrm{Diff}(M)$ such that $\varphi(0) = \varphi_0$ and $\varphi(1) = \varphi_1$. The *isotopy group* $\mathcal{S}(M \times I)$ is defined to be the fiber $\sigma^{-1}(\pi_I)$, i.e.

$$\mathcal{S}(M \times I) = \{ \ \bar{\varphi} \in \mathrm{Diff}(M \times I, M \times \{0\}) \mid \pi_I \circ \bar{\varphi} = \pi_I \ \}.$$

Clearly $\mathcal{S}(M \times I)$ is indeed a subgroup of $\mathrm{Diff}(M \times I, M \times \{0\})$. There is a natural action

$$\mu : \mathcal{S}(M \times I) \times \mathrm{Diff}(M) \longrightarrow \mathrm{Diff}(M)$$

defined as follows. Let $\bar{\psi} : (x,t) \mapsto (\psi_t(x), t)$ be an isotopy, and $\varphi \in \mathrm{Diff}(M)$. Then

$$\mu(\bar{\psi}, \varphi) = \varphi \circ \psi_1.$$

The action μ extends to the action

$$\tilde{\mu} : \mathrm{Diff}(M \times I, M \times \{0\}) \times \mathrm{Diff}(M) \longrightarrow \mathrm{Diff}(M)$$

which sends a pair $(\bar{\varphi}, \varphi)$, $\bar{\varphi} : M \times I \longrightarrow M \times I$ and $\varphi : M \to M$, to the diffeomorphism

$$\varphi \circ \left(\bar{\varphi}|_{M \times \{1\}}\right) : M \to M.$$

Then two diffeomorphisms $\varphi_0, \varphi_1 \in \mathrm{Diff}(M)$ are said to be *pseudo-isotopic* if there exists a pseudo-isotopy $\bar{\varphi} \in \mathrm{Diff}(M \times I, M \times \{0\})$ such that $\tilde{\mu}(\bar{\varphi}, \varphi_0) = \varphi_1$, i.e.

$$\varphi_0 \circ \left(\bar{\varphi}|_{M \times \{1\}}\right) = \varphi_1.$$

By construction, two isotopic diffeomorphisms are pseudo-isotopic. The converse does not hold, in general. Clearly the obstruction is the group of path-components

$$\pi_0(\mathrm{Diff}(M \times I, M \times \{0\}) \cong \pi_0(\mathcal{E}(M \times I)) \cong \pi_1(\mathcal{F}(M \times I), \mathcal{E}(M \times I)).$$

The following fundamental result is proven by J. Cerf:

Theorem 3.6. (J. Cerf, [7]) *Let M be a closed simply connected manifold of dimension* $\dim M \geq 5$. *Then*

$$\pi_0(\mathrm{Diff}(M \times I, M \times \{0\}) = \pi_1(\mathcal{F}(M \times I), \mathcal{E}(M \times I)) = 0.$$

In particular, any two pseudo-isotopic diffeomorphisms $\varphi_0, \varphi_1 \in \mathrm{Diff}(M)$ *are isotopic.*

In the case when a manifold M has non-trivial fundamental group, the group

$$\pi_0(\mathrm{Diff}(M \times I, M \times \{0\})$$

is identified with a corresponding Whitehead group $\mathrm{Wh}(\pi)$ which depends on the fundamental group $\pi = \pi_1 M$. The Whitehead group $\mathrm{Wh}(\pi)$ plays a fundamental role in smooth and geometric topology, see (say, survey [20]). In particular, the obstruction group is “almost always” non-zero if the fundamental group $\pi_1 M$ contains torsison. Otherwise, it is an open question whether the Whitehead group $\mathrm{Wh}(\pi)$ is nontrivial or not for a torison-free group π (see, say, [25, Conjecture 3.4]).

Let $\mathcal{C}(M \times I)$ be the space of conformal classes on $M \times I$. The pseudo-isotopy group $\mathrm{Diff}(M \times I, M \times \{0\})$ acts on the space of metrics $\mathcal{R}\mathrm{iem}(M \times I)$ and the space of conformal classes $\mathcal{C}(M \times I)$ by pull-back:

$$(\bar{\varphi}, \bar{g}) \mapsto \varphi^* \bar{g}, \qquad (\varphi, \bar{C}) \mapsto \bar{\varphi}^* \bar{C}.$$

In particular, if $\bar{\varphi} \in \mathrm{Diff}(M \times I, M \times \{0\})$, and $(M \times I, \bar{g})$ is a psc-concordance (respectively, $(M \times I, \bar{C})$ is a conformal psc-concordance), then $(M \times I, \bar{\varphi}^* \bar{g})$ is also psc-concordance (respectively, $(M \times I, \bar{\varphi}^* \bar{C})$ is a conformal psc-concordance).

3.5. **Isotopy and pseudo-isotopy of diffeomorphisms versus psc-concordance.** Let us return to a conformal psc-concordance $(M \times I, \bar{C})$. We choose a metric $\bar{g} \in \bar{C}$ with zero mean curvature along the boundary. Then we choose a slicing function $\bar{\alpha} \in \mathcal{E}(M \times I)$ and construct a smooth tangent vector field

$$X_{\bar{g}}(\bar{\alpha}) = \frac{\nabla_{\bar{g}} \bar{\alpha}}{|\nabla_{\bar{g}} \bar{\alpha}|^2_{\bar{g}}}.$$

It is easy to see that

$$d\bar{\alpha}(X_{\bar{g}}(\bar{\alpha})) = \bar{g} \left\langle \nabla_{\bar{g}} \bar{\alpha}, X_{\bar{g}}(\bar{\alpha}) \right\rangle = 1 \ .$$

We denote by γ_x the integral curve of the vector field $X_{\bar{g}}(\bar{\alpha})$ such that $\gamma_x(0) = (x, 0)$. It is easy to see that $\gamma_x(1) \in M \times \{1\}$.

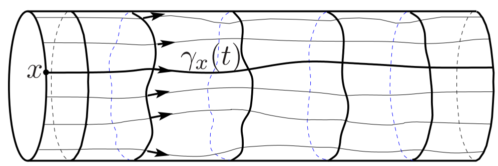

FIGURE 6. The integral curves $\gamma_x(t)$.

We obtain a diffeomorphism $\bar{\varphi} : M \times I \to M \times I$ defined by the formula

$$\bar{\varphi} : (x, t) \mapsto (\pi_M(\gamma_x(t)), \pi_I(\gamma_x(t)),$$

where $\pi_M : M \times I \to M$ and $\pi_I : M \times I \to I$ are the natural projections on the corresponding factors. Clearly $\bar{\varphi}|_{M \times \{0\}} = Id_M$, thus $\bar{\varphi} \in \mathrm{Diff}(M \times I, M \times \{0\})$ is a pseudo-isotopy.

By construction, we have that $\pi_I(\gamma_x(t)) = \bar{\alpha}(x, t)$. We introduce new coordinates:

$$(y, s) := (\pi_M(\gamma_x(t)), \pi_I(\gamma_x(t)).$$

Now we denote by $\tilde{g}(y, s)$ the metric $(\bar{\varphi}^{-1})^* \bar{g}(y, s)$. We have:

$$\begin{array}{rcl} \tilde{g}(y,s) & = & (\bar{\varphi}^{-1})^* \bar{g}(y,s) \\ & = & \bar{g}|_{M_s}(y) + \dfrac{1}{|\nabla_{\bar{g}} \bar{\alpha}|^2_{\bar{g}}} ds^2 \\ & = & \dfrac{1}{|\nabla_{\bar{g}} \bar{\alpha}|^2_{\bar{g}}} \left(|\nabla_{\bar{g}} \bar{\alpha}|^2_{\bar{g}} \cdot \bar{g}|_{M_s}(y) + ds^2 \right). \end{array}$$

We observe that $|\nabla_{\bar{g}} \bar{\alpha}|^2_{\bar{g}} \cdot \tilde{g} \in (\bar{\varphi}^{-1})^* \bar{C}$. Now we can replace the original conformal class $\bar{C}$ by the pull-back class $(\bar{\varphi}^{-1})^* \bar{C}$, and the metric $\bar{g}$ by the metric $|\nabla_{\bar{g}} \bar{\alpha}|^2_{\bar{g}} \cdot \tilde{g}$; i.e., we change the notations as follows:

$$\begin{array}{rcl} (y,s) & \rightsquigarrow & (x,t), \\ \bar{g} & \rightsquigarrow & |\nabla_{\bar{g}} \bar{\alpha}|^2_{\bar{g}} \cdot \tilde{g}, \\ |\nabla_{\bar{g}} \bar{\alpha}|^2_{\bar{g}} \cdot \bar{g}|_{M_s}(y) & \rightsquigarrow & g_t, \\ \bar{C} & \rightsquigarrow & (\bar{\varphi}^{-1})^* \bar{C}. \end{array}$$

It is easy to see that the resulting metric $\bar{g}$ has zero mean curvature along the boundary $M_0 \sqcup M_1$. Indeed, we have started with a metric which is minimal along the boundary, and the pseudoisotopy we have applied preserves minimality since the slicing function $\bar{\alpha}$ determining the pseudoisotopy agrees with the projection $\pi_I : M \times I \to I$ near the boundary $M_0 \sqcup M_1$. We summarize the above observations:

Proposition 3.7. (K. Akutagawa) *Let $\bar{C} \in \mathcal{C}(M \times I)$ be a conformal class, and $\bar{\alpha} \in \mathcal{E}(M \times I)$ be a slicing function. Then there exists a metric $\bar{g} \in \bar{C}$ with minimal boundary condition such that*

$$\begin{cases} \bar{g} & = & \bar{g}|_{M_t} + dt^2 \quad \text{on} \quad M \times I \\ \mathrm{Vol}_{g_t}(M_t) & = & \mathrm{Vol}_{g_0}(M_0) \quad \text{for all} \quad t \in I \end{cases}$$

up to a pseudo-isotopy given by the slicing function $\bar{\alpha}$.

3.6. **Smooth convergence of Riemannian manifolds.** Here we review basic facts on a compactness of the space of Riemannian manifolds in Gromov-Hausdorff topology. Let X be a smooth manifold (not necessary compact), and $x \in X$ be a base point. We recall that a sequence of compact sets $\{K_j\}$, $K_j \subset X$, *exhaust* the space X if for any compact subset $C \subset X$ there exists j_0 such that $C \subset K_j$ for $j \geq j_0$.

Let (X, h, x) be a *pointed* smooth, complete Riemannian manifold, i.e. $x \in X$ is a base point, and h is a complete Riemannian metric. We assume that the manifold X is also equipped with a reference Riemannian metric $\mathfrak{m}$.

Definition 3.8. Let $\{(X_i, h_i, x_i)\}$ be a sequence of such manifolds. The sequence $\{(X_i, h_i, x_i)\}$ is said to smoothly converge to (X, h, x) as $i \to \infty$ if there exist

- **(1)** a sequence of compact sets $\{K_i\}$, $K_i \subset X$, exhausting the space X with $x \in K_i$ for each i,
- **(2)** a sequence of smooth maps $\varphi_i : K_i \to X_i$ which are diffeomorphisms on their image satisfying $\varphi_i(x) = x_i$ for each i,

such that the sequence of metrics $\{\varphi_i^*(h_i)\}$ smoothly converges to the metric h, i.e., the sequence of tensors $\{\varphi_i^*(h_i) - h\}$ and all its derivatives (with respect to some reference Riemannian metric $\mathfrak{m}$) converge to zero uniformly on every compact subset $C \subset X$.

Let $B_{h_i}(x_i, \rho) \subset X_i$ be the ball of radius $\rho > 0$ centered at x_i. We denote also by $\mathrm{inj}(X_i, h_i, x_i)$ the injectivity radius at the point x_i. It is well-known that a smooth convergence $(X_i, h_i, x_i) \to (X, h, x)$ implies the following geometrical bounds:

$$\sup_i \sup_{x \in B_{h_i}(x_i,\rho)} |\nabla_{\mathfrak{m}}^k \mathrm{Rm}_{h_i}(x)| < \infty \quad \text{for each } \rho > 0 \text{ and integer } k \geq 0, \tag{9}$$

$$\inf_i \mathrm{inj}(X_i, h_i, x_i) < \infty \ . \tag{10}$$

Various versions of the following result (see, say [30, Theorem 7.1.3]) appear in, or can be derived from, papers of (for example) Greene and Wu [14], Fukaya [13], all of which can be traced back to original ideas of Gromov and Cheeger [15]:

Theorem 3.9. (Gromov-Cheeger) *Let* (X_i, h_i, x_i) *with* $\dim X = n$ *be a sequence of pointed, complete Riemannian manifolds satisfying* (9) *and* (10). *Then there exists a subsequence* $(X_{i_s}, h_{i_s}, x_{i_s})$ *of* (X_i, h_i, x_i) *which smoothly converges to a pointed, complete Riemannian manifold* (X, h, x) *with* $\dim X = n$.

3.7. **Example.** Let $(M \times I, \bar{g})$ be a psc-concordance as above. We choose the projection $\pi_I : M \times I \to I$ on the second factor as a slicing function, i.e. $\bar{\alpha} = \pi_I$. We assume that the metric $\bar{g}$ is given as $\bar{g} = g_t + dt^2$ with respect to the slicing function $\bar{\alpha} = \pi_I$. Then for every pair $0 \leq t < s \leq 1$, we have the Riemannian manifolds $(W^*_{t,s}, \bar{g}^*_{t,s})$, where

$$W^*_{t,s} = \bar{\alpha}^{-1}([t, s]), \quad \bar{g}^*_{t,s} = \bar{g}|_{W_{t,s}}$$

and the boundary $\partial W^*_{t,s} = M_t \sqcup M_s$ is equipped with the metrics $g_\mu = \bar{g}|_{M_\mu}$, $\mu = t, s$. Furthemore, since the slicing function $\bar{\alpha}$ coincides with the projection, we may identify the manifold $W_{t,s}$ with the product $M \times [t, s]$.

We choose a base point $x_0 \in M_0$, and denote by $x_s \in M_s$ the base point given as $x_0 = \pi_M(x_s)$, where $\pi_M : M \times I \to M$ is the projection on the first factor. Now let $I_k = [t_k, s_k] \subset [0, 1]$ be a sequence of nested intervals:

$$[0, 1] \supset I_1 \supset I_2 \supset \cdots I_k \supset I_{k+1}, \quad \cap_k I_k = \{s^o\}.$$

In particular, $s^o \in I_k$ for each k, and $t_k \to s^o$ and $s_k \to s^o$ as $k \to \infty$. We assume that $x_{s^o} \in M_{s^o} \subset W_{t_k,s_k}$ is a base point for each k.

Let $\xi_{t,s} : [0, 1] \to [t, s]$ be a linear homeomorphism, where $\tau \mapsto (1 - \tau)t + \tau s$. We define a diffeomorphism

$$\bar{\xi}_{t,s} : M \times [0, 1] \to W_{t,s} \cong M \times [t, s],$$

as $\bar{\xi}_{s,t} : (x, \tau) \mapsto (x, \xi_{s,t}(\tau))$.

We define the metrics $\bar{g}_{t,s} = \bar{\xi}^*_{t,s}(\bar{g}^*_{t,s})$ on $M \times I$, let $x_k = (x_0, \tau(s^o)) \in M \times I$ be the base point, and consider the sequence of Riemannian manifolds

$$\{(M \times I, \bar{g}_{t_k,s_k}, x_k)\}. \tag{11}$$

The following lemma is straightforward.

Lemma 3.10. *For the sequence* (11) *of Riemannian manifolds the conditions* (9) *and* (10) *are satisfied. Moreover, there exists a smooth limit* $(W^o, \bar{g}^o, x^o)$ *where*

$$W^o \cong M \times I, \quad \bar{g}^o = g_{s^o} + dt^2, \tag{12}$$

i.e. $(W^o, \bar{g}^o, x^o)$ *could be identified with the cylindrical manifold* $(M \times I, g_{s^o} + dt^2, x^o)$.

3.8. Λ**-function associated to a conformal psc-concordance.** Consider a conformal psc-concordance $(M \times I, \bar{C})$, and choose a metric $\bar{g} \in C$ with minimal boundary condition. For a given slicing function $\bar{\alpha} : M \times I \to I$, for each pair $t, s \in I$, $t < s$, we consider the Riemannian manifold $(W^*_{t,s}, \bar{g}^*_{t,s})$, where

$$W^*_{t,s} = \bar{\alpha}^{-1}([t, s]), \quad \bar{g}^*_{t,s} = \bar{g}|_{W^*_{t,s}}.$$

Now we let $\xi_{t,s} : [0,1] \to [t,s]$ be a linear function sending $\tau \mapsto (1-\tau)t + \tau s$. This gives a diffeomorphism

$$\bar{\xi}_{t,s} : M \times [0,1] \to M \times [s,t], \qquad (x,\tau) \mapsto (x, \xi_{t,s}(\tau)).$$

We use the map $\bar{\xi}_{t,s}$ to stretch the manifold $(W^*_{t,s}, \bar{g}^*_{t,s})$ in the horizontal direction, and denote by $(M \times [0,1], \bar{g}_{t,s})$ the resulting stretched manifold, see Fig. 7. Then $\partial(M \times [0,1]) = M_t \sqcup M_s$, where $M_\tau = \bar{\alpha}^{-1}(\tau)$, $\tau = t, s$. We denote by $h_{\bar{g}_{t,s}} = \frac{1}{n-1} \operatorname{tr} A_{\bar{g}_{t,s}}$ the

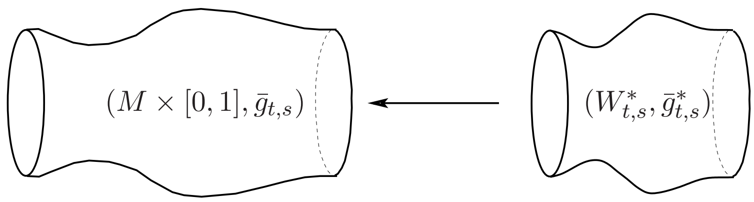

FIGURE 7. Stretching $(W^*_{t,s}, \bar{g}^*_{t,s})$ to get $(M \times [0,1], \bar{g}_{t,s})$.

normalized mean curvature along the boundary $\partial(M \times [0,1]) = M_t \sqcup M_s$. Let

$$L_{\bar{g}_{t,s}} = a_n \Delta_{\bar{g}_{t,s}} + R_{\bar{g}_{t,s}}, \quad \text{where} \quad a_n = \tfrac{4(n-1)}{n-2},$$

be the conformal Laplacian on the manifold $(M \times [0,1], \bar{g}_{t,s})$. We denote by $\lambda_1(L_{\bar{g}_{t,s}})$ the principal eigenvalue of the conformal Laplacian $L_{\bar{g}_{t,s}}$ on $(M \times [0,1], \bar{g}_{t,s})$ with the minimal boundary condition. We obtain a function

$$\Lambda_{(M\times I, \bar{g}, \bar{\alpha})} : (t,s) \mapsto \lambda_1(L_{\bar{g}_{t,s}}).$$

Definition 3.11. The function $\Lambda_{(M\times I, \bar{g}, \bar{\alpha})}$ is called Λ*-function associated to the triple* $(M \times I, \bar{g}, \bar{\alpha})$. We say that the triple $(M \times I, \bar{g}, \bar{\alpha})$ is *non-negative* if its Λ-function is non-negative.

We denote by T and $\bar{T}$ the half-closed and closed triangles

$$T = \{ \ (s,t) \mid 0 \le s < t \le 1 \ \}, \qquad \bar{T} = \{ \ (s,t) \mid 0 \le s \le t \le 1 \ \}.$$

Proposition 3.12. *The function $\Lambda_{(M\times I, \bar{g}, \bar{\alpha})}$ satisfies the following properties:*

(a) *$\Lambda_{(M\times I, \bar{g}, \bar{\alpha})}$ is continuous on T and there is a unique continuous extension $\bar{\Lambda}_{(M\times I, \bar{g}, \bar{\alpha})} : \bar{T} \to \mathbf{R}$;*
(b) *there exists a family of smooth positive functions $u_{t,s}$ on $W_{t,s}$ such that for each $(t,s) \in T$, the function $u_{t,s}$ is a normalized eigenfunction of $L_{\bar{g}_{t,s}}$ corresponding to a principal eigenvalue with minimal boundary condition, and the family $u_{t,s}$ continuously depends on (t,s).*

Clearly **(a)** and **(b)** imply that the limit $\bar{\Lambda}_{(M\times I, \bar{g}, \bar{\alpha})}(s) = \lim\limits_{t\to s} \Lambda_{t,s}$ exists or all $s \in I$.

4. Proof of Theorem 2.8

4.1. **Almost conformal Laplacian.** Let M be a closed manifold as above with $\dim M = n-1 \geq 3$. For a given Riemannian metric g on M, we consider the elliptic operator

$$\mathcal{L}_g = a_n \Delta_g + R_g, \quad \text{where} \quad a_n = \tfrac{4(n-1)}{n-2}.$$

We call $\mathcal{L}_g$ *almost conformal Laplacian*, see [3, Section 2]: it is slightly different from the conformal Laplacian $L_g = a_{n-1}\Delta_g + R_g$. Since

$$a_{n-1} - a_n = \tfrac{4}{(n-3)(n-2)}$$

the difference $\mathcal{L}_g - L_g$ is a positive operator. This implies the following fact (see, say [3, Lemma 2.10]). The proof is straightforward.

Lemma 4.1. *Let $\lambda_1(\mathcal{L}_g)$ be the principal eigenvalue of the operator $\mathcal{L}_g$.*

(i) *If $\lambda_1(\mathcal{L}_g) > 0$, then $\lambda_1(L_g) > 0$. Thus $\lambda_1(\mathcal{L}_g) > 0$ implies that $[g] \in \mathcal{C}^+(M)$.*

(ii) *If $\lambda_1(\mathcal{L}_g) = 0$. Then either $R_g \equiv 0$, or there exits a metric $\tilde{g} \in [g]$ such that $R_{\tilde{g}} > 0$. Thus $\lambda_1(\mathcal{L}_g) = 0$ implies that $[g] \in \mathcal{C}^{\geq 0}(M)$.*

Now we can return to the proof of Theorem 2.8.

4.2. **Proof of Theorem 2.8 in a special case.** Let $(M \times I, \bar{g})$ be a psc-concordance. In addition to the assumption of Theorem 2.8, we assume that

(1) the slicing function $\bar{\alpha}$ coincides with the projection $\pi_I : M \times I \to I$;

(2) the metric $\bar{g}$ is given as $\bar{g} = g_t + dt^2$.

We choose $s^o \in I$ and a nested sequence of intervals $\{[t_k, s_k]\}$ such that $t_k \to s^o$ and $s_k \to s^o$ as $k \to \infty$. Then we have the manifolds $(W^*_{t_k,s_k}, \bar{g}_{t_k,s_k})$, where

$$W^*_{t_k,s_k} = \bar{\alpha}^{-1}([t_k, s_k]), \quad \bar{g}^*_{t_k,s_k} = \bar{g}|_{W^*_{t_k,s_k}}.$$

Then we use a stretching as above to construct the manifolds $(M \times I, \bar{g}_{t_k,s_k})$, and denote by $\lambda_1(L_{\bar{g}_{t_k,s_k}})$ the principal eigenevalue of the minimal boundary problem on the manifold $(M \times I, \bar{g}_{t_k,s_k})$. We obtain a sequence of Riemannian manifolds $(M \times I, \tilde{g}_{t,s}, x_k)$ as in (11). Recall that $\bar{g}_{s,t} = \bar{\xi}^*_{s,t}(\bar{g}^*_{s,t})$ is the metric on $M \times I$, where

$$\bar{\xi}_{s,t} : M \times [0,1] \to W^*_{s,t} \cong M \times [s,t]$$

is the diffeomorphism given by the linear map $\tau \mapsto (1-\tau)s + \tau t$. Then by Lemma 3.10, the sequence $(M \times I, \bar{g}_{t_k,s_k}, x_k)$ converges to the cylindrical manifold $(M \times I, g_{s^o} + dt^2)$ and by construction,

$$\lambda_1(L_{\bar{g}_{t_k,s_k}}) \to \lambda_1(L_{g_{s^o}+dt^2}),$$

where $\lambda_1(L_{g_{s^o}+dt^2})$ is the principal eigenvalue of the minimal boundary problem on the cylindrical manifold $(M \times I, g_{s^o} + dt^2)$. It is easy to see that

$$\lambda_1(L_{g_{s^o}+dt^2}) = \lambda_1(\mathcal{L}_{g^o_s}),$$

where $\mathcal{L}_{g_s^*}$ is the almost conformal Laplacian on (M_{s^o}, g_{s^o}). According to Proposition 3.12, the limit

$$\lim_{k\to\infty} \lambda_1(L_{\bar{g}_{t_k,s_k}})$$

exists, and by the assumption, $\lambda_1(L_{\bar{g}_{t_k,s_k}}) \geq 0$. Thus $\lambda_1(\mathcal{L}_{g_s^*}) \geq 0$ for each $s^o \in [0,1]$.

According to Lemma 4.1, the conformal classes $C_t = [g_t]$ are all non-negative. Then the Ricci flow argument given in Section 2.7 completes the proof.

4.3. **The general case.** Let $(M \times I, \bar{C})$ be a conformal psc-concordance between positive conformal classes C_0 and C_1, and $\bar{g} \in \bar{C}$ be a metric with zero mean curvature along the boundary. Assume that $\Lambda_{(M\times I,\bar{g},\bar{\alpha})} \geq 0$ for a given slicing function $\bar{\alpha} : M \times I \to I$.

Then, according to section 3.5, namely, Proposition 3.7, we may assume that up to pseudo-isotopy, our metric $\bar{g}$ is given as $\bar{g} = g_t + dt^2$, and the slicing function $\bar{\alpha}$ coincides with the projection $\pi_I : M \times I \to I$. This is exactly the "special case" that we just have proved above. This concludes our proof of Theorem 2.8. □

5. Connected sum of scalar-flat manifolds

5.1. **Introduction.** We are getting ready to introduce a construction which we call a *bypass surgery*: this is key in proving Theorem 2.9. However, to make it work, we use the results and constructions of surgery on scalar-flat manifolds due to L. Mazzieri [23]. There is also a result by D. Joyce [17] concerning a connected sum construction. We review some of those results, where we use the notations which are convenient to us.

5.2. **The case when a manifold has empty boundary.** Let $(W_1, \bar{g}_1)$ and $(W_2, \bar{g}_2)$ be two compact manifolds of the same dimension $n \geq 4$ with scalar-flat metrics $\bar{g}_1$ and $\bar{g}_2$. For now we assume that $\partial W_1 = \emptyset = \partial W_2$. We assume also that it is given a Riemannian manifold (V, h_V) together with isometric embeddings

$$\iota_1 : V \hookrightarrow W_1, \quad \iota_2 : V \hookrightarrow W_2.$$

Moreover, we assume that both embeddings are such that V has trivial normal bundle in W_1 and in W_2, where q is a codimension of V. We fix such a neighbourhood $U = V \times D^q$ and its embeddings extending ι_1 and ι_2.

$$\bar{\iota}_1 : U \hookrightarrow W_1, \quad \bar{\iota}_2 : U \hookrightarrow W_2.$$

In this setting, we can glue the manifolds W_1 and W_2 along the submanifold V, i.e. we get a manifold $W := W_1 \cup_U W_2$. Here is the first result by L. Mazzieri:

Theorem 5.1. (See [23, Theorem 1]) *Let $(W_1, \bar{g}_1)$ and $(W_2, \bar{g}_2)$ be two compact Riemannian manifolds with $\partial W_1 = \emptyset$ and $\partial W_2 = \emptyset$, and (V, h_V) be a submanifold of codimension $q \geq 3$. Assume that $R_{\bar{g}_1} \equiv 0$, $R_{\bar{g}_2} \equiv 0$ and $\mathrm{Vol}_{\bar{g}_1}(W_1) = \mathrm{Vol}_{\bar{g}_1}(W_1) = 1$.*

Then there exists $\varepsilon_0 > 0$ and a family of metrics $\bar{g}_\varepsilon$ on $W = W_1 \cup_U W_2$, $0 < \varepsilon < \varepsilon_0$, such that

- **(i)** *$\bar{g}_\varepsilon|_{W_i \setminus U} \to \bar{g}_i|_{W_i\setminus U}$ in C^2-topology on compact sets of $W_i \setminus V$, $i = 1, 2$;*

(ii) $\bar{g}_\varepsilon|_{W_i\setminus U}$ *is conformal to* $\bar{g}_i|_{W_i\setminus U}$ *away from a small neighbourhood of* V, *i.e. there is a conformal metric* $\tilde{g}_\varepsilon = u_\varepsilon^{\frac{4}{n-2}}\bar{g}_\varepsilon$, *such that*

$$\tilde{g}_\varepsilon|_{W_i\setminus U} = \bar{g}_i|_{W_i\setminus U}, \quad i = 1, 2,$$

with the conformal factor u_ε *satisfying*

$$\|u_\varepsilon - 1\|_{L^\infty} \leq C \cdot \varepsilon^\gamma,$$

where $C > 0$ *and* $\gamma \in (0, 1/4)$;

(ii) $R_{\bar{g}_\varepsilon} = \mathcal{O}(\varepsilon^{n-2})$.

Remark. L. Mazzieri works in a more general setting: he does not requires the submanifold V to have a trivial normal bundle. However, this is enough for our goals.

The second result gives a gluing construction of two scalar-flat metrics:

Theorem 5.2. (See [23, Theorem 2]) *Let* $(W_1, \bar{g}_1)$ *and* $(W_2, \bar{g}_2)$ *be two compact Riemannian manifolds with* $\partial W_1 = \emptyset$ *and* $\partial W_2 = \emptyset$, *and* (V, h_V) *be a submanifold of codimension* $q \geq 3$. *Assume that the scalar-flat metrics* $\bar{g}_1$ *and* $\bar{g}_2$ *are not Ricci-flat.*

Then there exists $\varepsilon_0 > 0$ *such that for every* ε, $0 < \varepsilon < \varepsilon_0$, *there exists a scalar-flat metric* $\bar{g}_\varepsilon$ *on* $W = W_1 \cup_U W_2$ *such that the restrictions* $\bar{g}_\varepsilon|_{W_1\setminus U}$, $\bar{g}_\varepsilon|_{W_2\setminus U}$ *converge to the corresponding metrics* $\bar{g}_1|_{W_1\setminus U}$, $\bar{g}_2|_{W_2\setminus U}$ *in* C^2*-topology on compact sets of* $W_1 \setminus V$ *and* $W_2 \setminus V$ *as* $\varepsilon \to 0$.

5.3. **The case of non-trivial boundary.** Now we consider the case when the manifolds W_1 and W_2 have non-empty boundaries. If the submanifold $V \subset W_1, W_2$ has no boundary and is located in the interior of the manifolds W_1, W_2, then it is easy to see that Theorem 5.1 extends to that case. The case we really need is one in which V also has a boundary that is transversal to the boundaries ∂W_1 and ∂W_2. We also assume that

$$\partial V = V \cap W_1, \qquad \partial V = V \cap W_2.$$

Again, we assume that both embeddings are such that V has a trivial normal bundle in W_1 and in W_2, where q is a codimension of V. We fix such a neighbourhood $U = V \times D^q$ and its embeddings extending ι_1 and ι_2.

$$\bar{\iota}_1 : U \hookrightarrow W_1, \qquad \bar{\iota}_2 : U \hookrightarrow W_2.$$

Here is the generalization of Theorem 5.1 we need:

Theorem 5.3. *Let* $(W_1, \bar{g}_1)$ *and* $(W_2, \bar{g}_2)$ *be two compact manifolds with* $\partial W_1 \neq \emptyset$ *and* $\partial W_2 \neq \emptyset$, *and* (V, h_V) *be a submanifold of codimension* $q \geq 3$. *Assume that* $R_{\bar{g}_1} \equiv 0$, $R_{\bar{g}_2} \equiv 0$ *and* $\mathrm{Vol}_{\bar{g}_1}(W_1) = \mathrm{Vol}_{\bar{g}_1}(W_1) = 1$, *and the mean curvature of the metrics* $\bar{g}_1$ *and* $\bar{g}_2$ *is identically zero along corresponding boundaries* ∂W_1 *and* ∂W_2.

Then there exists $\varepsilon_0 > 0$ *and a family of metrics* $\bar{g}_\varepsilon$ *on* $W = W_1 \cup_U W_2$, $0 < \varepsilon < \varepsilon_0$, *such that*

(i) $\bar{g}_\varepsilon|_{W_i\setminus U} \to \bar{g}_i|_{W_i\setminus U}$ *in* C^2*-topology on compact sets of* $W_i \setminus V$, $i = 1, 2$;

(ii) $\bar{g}_\varepsilon|_{W_i \setminus U}$ *is conformal to* $\bar{g}_i|_{W_i\setminus U}$ *away from a small neighbourhood of* V*, i.e. there is a conformal metric* $\tilde{g}_\varepsilon = u_\varepsilon^{\frac{4}{n-2}} \bar{g}_\varepsilon$*, such that*

$$\tilde{g}_\varepsilon|_{W_i\setminus U} = \bar{g}_i|_{W_i\setminus U}, \quad i = 1, 2,$$

with the conformal factor u_ε *satisfying*

$$\|u_\varepsilon - 1\|_{L^\infty} \le C \cdot \varepsilon^\gamma,$$

where $C > 0$ *and* $\gamma \in (0, 1/4)$*;*

(ii) $R_{\bar{g}_\varepsilon} = \mathcal{O}(\varepsilon^{n-2})$*.*

(iii) $H_{\bar{g}_\varepsilon} = \mathcal{O}(\varepsilon^{n-2})$ *along the boundary* $\partial W = \partial W_1 \cup_{\partial V} \partial W_2$*.*

Proof of Theorem 5.3 is a straightforward generalization of the proof of Theorem 5.1 given by L. Mazzieri. Theorem 5.3 is enough for our purposes. However, it is reasonable to suggest that the following generalization holds:

Conjecture 5.4. *Let* $(W_1, \bar{g}_1)$*,* $(W_2, \bar{g}_2)$ *be two compact manifolds with non-empty boundaries* $N_1 = \partial W_1$*,* $N_2 = \partial W_2$*, where* $\bar{g}_1$ *and* $\bar{g}_2$ *are scalar-flat metrics with minimal boundary condition. Assume in addition that the scalar-flat metrics* $\bar{g}_1$ *and* $\bar{g}_2$ *are not Ricci-flat. Let* (V, h_V) *be a Riemannian manifold given together with isometric embeddings*

$$\iota_1 : V \hookrightarrow W_1, \qquad \iota_2 : V \hookrightarrow W_2,$$

embedded with a trivial normal bundle $U = V \times D^q$*, where the codimension* $q \ge 3$*. Besides, we assume that the boundary* V *is transversal to* ∂W_1 *and* ∂W_2*, and*

$$\partial V = V \cap W_1, \qquad \partial V = V \cap W_2.$$

Then there exists $\varepsilon_0 > 0$ *such that for every* ε*,* $0 < \varepsilon < \varepsilon_0$*, there exists a scalar-flat metric* $\bar{g}_\varepsilon$ *on* $W = W_1 \cup_U W_2$ *with minimal boundary condition such that the restrictions* $\bar{g}_\varepsilon|_{W_1\setminus U}$*,* $\bar{g}_\varepsilon|_{W_2\setminus U}$ *converge to the corresponding metrics* $\bar{g}_1|_{W_1\setminus U}$*,* $\bar{g}_2|_{W_2\setminus U}$ *in* C^2*-topology on compact sets of* $W_1 \setminus V$ *and* $W_2 \setminus V$ *as* $\varepsilon \to 0$*.*

6. Two boundary problems for the conformal Laplacian

6.1. **The boundary problems.** Let $(W, \bar{g})$ be a Riemannian manifold with non-empty boundary $(\partial W, g)$. As above, we denote by $A_{\bar{g}}$ the second fundamental form along ∂W, by $H_{\bar{g}} = \operatorname{tr} A_{\bar{g}}$ the mean curvature along ∂W, and by $h_{\bar{g}} = \frac{1}{n-1} H_{\bar{g}}$ the "normalized" mean curvature. Also we denote by ∂_ν the directional derivative with respect to an outward unit normal vector field along the boundary ∂W.

Then for a conformal metric $\tilde{g} = u^{\frac{4}{n-2}}\bar{g}$, we have the following standard formulas for the scalar and mean curvatures:

$$\begin{array}{rcl} R_{\tilde{g}} &=& u^{-\frac{n+2}{n-2}}\left(\frac{4(n-1)}{n-2}\Delta_{\bar{g}} u + R_{\bar{g}} u\right) \;=\; u^{-\frac{n+2}{n-2}} L_{\bar{g}} u, \\ h_{\tilde{g}} &=& \frac{2}{n-2} u^{-\frac{n}{n-2}}\left(\partial_\nu u + \frac{n-2}{2} h_{\bar{g}} u\right) \;=\; u^{-\frac{n}{n-2}} B_{\bar{g}} u. \end{array} \tag{13}$$

The first boundary problem on $(W,\bar{g})$ is the *minimal boundary problem* given as

$$\left\{\begin{array}{rcll} L_{\bar{g}}u & = & \frac{4(n-1)}{n-2}\Delta_{\bar{g}}u + R_{\bar{g}}u = \lambda_1 u & \text{on } W, \\ & & & \\ B_{\bar{g}}u & = & \partial_\nu u + \frac{n-2}{2}h_{\bar{g}}u = 0 & \text{on } \partial W. \end{array}\right. \tag{14}$$

Here λ_1 is the *principal eigenvalue* corresponding to the minimal boundary problem. Recall that the eigenvalue λ_1 could be defined as the Rayleigh quotient

$$\lambda_1 = \inf_{f\in C^\infty_+} \frac{\int_W (a_n|\nabla_{\bar{g}}f|^2 + R_{\bar{g}}f^2)d\sigma_{\bar{g}} + 2(n-1)\int_{\partial W} h_{\bar{g}}f^2 d\sigma g}{\int_W f^2 d\sigma_{\bar{g}}} \tag{15}$$

If u is the eigenfunction corresponding to the principal eigenvalue, i.e. $L_{\bar{g}}u = \lambda_1 u$, $B_{\bar{g}}u = 0$, and $\tilde{g} = u^{\frac{4}{n-2}}\bar{g}$, then

$$\left\{\begin{array}{rcll} R_{\tilde{g}} & = & u^{-\frac{n+2}{n-2}}L_{\bar{g}}u = \lambda_1 u^{-\frac{4}{n-2}} & \text{on } W \\ & & & \\ h_{\tilde{g}} & = & u^{-\frac{n}{n-2}}B_{\bar{g}}u = 0 & \text{on } \partial W. \end{array}\right. \tag{16}$$

Clearly, the conformal metric $\tilde{g}$ is minimal along the boundary and has a definite sign of scalar curvature on the entire manifold W. It is well-known that the sign of the eigenvalue λ_1 is conformally invariant, i.e., it depends only on the conformal class $\bar{C} = [\bar{g}]$ of the metric $\bar{g}$ (see [11, Proposition 1.4]).

Definition 6.1. A conformal manifold $(W,\bar{C})$ is *positive* (respectively, *zero* or *negative*) if the eigenvalue λ_1 is positive (respectively, zero or negative).

The second boundary problem on $(W,\bar{g})$ is the *scalar-flat boundary problem* given as

$$\left\{\begin{array}{rcll} L_{\bar{g}}v & = & \frac{4(n-1)}{n-2}\Delta_{\bar{g}}v + R_{\bar{g}}v = 0 & \text{on } W, \\ & & & \\ B_{\bar{g}}v & = & \partial_\nu v + \frac{n-2}{2}h_{\bar{g}}v = \mu_1 v & \text{on } \partial W. \end{array}\right. \tag{17}$$

Here μ_1 is the *principal eigenvalue* corresponding to the scalar-flat boundary problem. Recall μ_1 could be defined through the Rayleigh quotient

$$\mu_1 = \inf_{f\in C^\infty_+} \frac{\int_W (a_n|\nabla_{\bar{g}}f|^2 + R_{\bar{g}}f^2)d\sigma_{\bar{g}} + 2(n-1)\int_{\partial W} h_{\bar{g}}f^2 d\sigma_{\bar{g}}}{\int_{\partial W} f^2 d\sigma_g}. \tag{18}$$

If v is the eigenfunction corresponding to the principal eigenvalue μ_1, i.e. $L_{\bar{g}}v = 0$, $B_{\bar{g}}v = \mu_1 v$, and $\hat{g} = v^{\frac{4}{n-2}}\bar{g}$, then

$$\left\{\begin{array}{rcll} R_{\hat{g}} & = & v^{-\frac{n+2}{n-2}}L_{\bar{g}}v = 0 & \text{on } W, \\ & & & \\ h_{\hat{g}} & = & v^{-\frac{n}{n-2}}B_{\bar{g}}v = \mu_1 v^{-\frac{2}{n-2}} & \text{on } \partial W. \end{array}\right. \tag{19}$$

The eigenfunction v is usually normalized as $\int_{\partial W} v^2 d\sigma_g = 1$. Thus the conformal metric $\hat{g}$ is sclar-flat and its mean curvature has a definite sign, the same as the principal eigenvalue μ_1. We recall the following basic fact:

Proposition 6.2. [11, Proposition 1.5] *The principal eigenvalue λ_1 of the boundary problem* (14) *and the principal eigenvalue μ_1 of the boundary problem* (17) *have the same sign. In particular, a conformal manifold $(W, \bar{C})$ is positive (respectively, zero or negative) if and only if the eigenevalue μ_1 is positive (respectively, zero or negative).*

We notice the following simple fact:

Lemma 6.3. *Let $(W, \bar{g})$ be a Riemannian manifold with a non-empty boundary $(\partial W, g)$. Assume v is a normalized positive eigenfunction solving the scalar-flat boundary problem* (17)*, so that $h_{\hat{g}} = \mu_1 v^{-\frac{2}{n-2}}$ for the conformal metric $\hat{g} = v^{\frac{4}{n-2}}\bar{g}$. Then*

$$\int_{\partial W} h_{\hat{g}} d\sigma_{\hat{g}|\partial W} = \mu_1.$$

Proof. Indeed, we have $\hat{g}|_{\partial W} = v^{\frac{4}{n-2}} g$, then $d\sigma_{\hat{g}|\partial W} = v^{\frac{2(n-1)}{n-2}} d\sigma_g$. Thus

$$\int_{\partial W} h_{\hat{g}} d\sigma_{\hat{g}|\partial W} = \mu_1 \int_{\partial W} v^{-\frac{2}{n-2}} \cdot v^{\frac{2(n-1)}{n-2}} d\sigma_g = \mu_1 \int_{\partial W} v^2 d\sigma_g = \mu_1$$

because of the normalizing constraint $\int_{\partial W} v^2 d\sigma_g = 1$. □

6.2. **Kobayashi metric.** Here we examine a particular manifold, the cylinder

$$(S^{n-1} \times I, \bar{\hbar}^{(\ell)}), \quad \bar{\hbar}^{(\ell)} = \hbar^{(\ell)} + dt^2,$$

where $\hbar^{(\ell)}$ is the Kobayashi metric defined below.

Proposition 6.4. (O. Kobayashi [19]) *Let ℓ be any positive integer and $\varepsilon > 0$. Then there exists a psc-metric $\hbar^{(\ell)}$ on the sphere S^{n-1} such that*

(a) $|R_{\hbar^{(\ell)}} - (\ell + 1)| < \varepsilon$,
(b) $\mathrm{Vol}_{\hbar^{(\ell)}}(S^{n-1}) = 1$.

Remark. We call the metric $\hbar^{(\ell)}$ *Kobayashi meric*. If we replace (a) by a weaker version:

(a′) $R_{\hbar^{(\ell)}} > \ell$,

then the metric $\hbar^{(\ell)}$ could be easily constructed by taking a connected sum of standard spheres; such metric is otherwise known as a "dumbbell metric". The condition (a′) will be enough for our purposes.

Lemma 6.5. *Let $\mathcal{L}_{\hbar} = a_n \Delta_{\hbar} + R_{\hbar}$ be the almost conformal Laplacian as above, and $C_0 > 0$ be a given constant. Then there exists $\ell \geq 2$ such that $\lambda_1(\mathcal{L}_{\hbar^{(\ell)}}) > C_0$, where $\lambda_1(\mathcal{L}_{\hbar^{(\ell)}})$ is the principal eigenevalue of $\mathcal{L}_{\hbar^{(\ell)}}$.*

Proof. Indeed, we may choose $\hbar^{(\ell)}$ such that $R_{\hbar^{(\ell)}} > \ell \geq 2$. Then the almost conformal Laplacian $\mathcal{L}_{\hbar^{(\ell)}} = a_n \Delta_{\hbar^{(\ell)}} + R_{\hbar^{(\ell)}}$ has the principal eigenvalue at least ℓ. The statement follows. □

Remark. The Kobayashi metric $\hbar^{(\ell)}$ plays an important role below. To simplify some estimates, everywhere below we will assume that

$$\lambda_1(\mathcal{L}_{\hbar^{(\ell)}}) > 16. \tag{20}$$

In the proof of Lemma 6.6 (see Apendix), we use a parameter k such that $a_n k^2 = \lambda_1(\mathcal{L}_{\hbar^{(\ell)}})$. Then we have to choose k such that $\frac{e^k}{e^{2k}+1} \leq \frac{1}{4}$ which holds if $k \geq 2$, which, in turn, holds if $\lambda_1(\mathcal{L}_{\hbar^{(\ell)}}) > 16$.

Lemma 6.6. *Let $(S^{n-1} \times I, \bar{\hbar}^{(\ell)})$, where $\bar{\hbar}^{(\ell)} = \hbar^{(\ell)} + dt^2$ be the cylinder. Let $C > 0$ be a given constant. Then for any ℓ satisfying $\frac{3}{4} a_n \lambda_1(\mathcal{L}_{\hbar^{(\ell)}}) > C^2$, there exists a metric $\tilde{\hbar}^{(\ell)} \in [\bar{\hbar}^{(\ell)}]$ such that*

(1) $R_{\tilde{\hbar}^{(\ell)}} \equiv 0$,
(2) $\mu_1(L_{\hbar^{(\ell)}}) = \int_{S^{n-1}\times\{0,1\}} h_{\tilde{\hbar}^{(\ell)}} d\sigma_{\tilde{\hbar}^{(\ell)}} \geq C$.

We provide a proof of Lemma 6.6 in Appendix.

6.3. **Remarks concerning the Kobayashi metric.** Let $(S^{n-1}, \hbar^{(\ell)})$ be a sphere with the Kobayashi metric as above. As it was mentioned, the metric $\hbar^{(\ell)}$ could be constructed

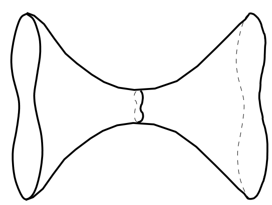

FIGURE 8. Scalar-flat metric $\tilde{\hbar}^{(\ell)}$.

by taking connected sum of standard spheres (via Gromov-Lawson construction, see Section 9): $(S^{n-1}, \hbar^{\ell}) = \#_{k(\ell)}(S^{n-1}, h_0)$. Next lemma follows from M. Walsh's work [31]:

Lemma 6.7. *Let (M, g) be a Riemannian manifold with psc-metric g, and*

$$(M, \hat{g}) := (M, g)\#(S^{n-1}, \hbar^{(\ell)})$$

be a connected sum given by Gromov-Lawson construction. (In particular, $\hat{g}$ is a psc-metric). Then the metrics g and $\hat{g}$ are psc-isotopic.

6.4. **Mean curvature function as an obstruction.** Let $(W, \tilde{g})$ be a manifold with non-empty boundary ∂W. Recall that we say that $(W, \tilde{g})$ is *of zero conformal class* if there exists a conformal metric $\bar{g} \in [\tilde{g}]$ such that $R_{\bar{g}} \equiv 0$ and the mean curvature $H_{\bar{g}} \equiv 0$ along the boundary ∂W. As before, we denote by $h_{\bar{g}}$ the normalized mean curvature. We observe that a conformal manifold of a zero conformal class cannot have an arbitrary mean curvature:

Lemma 6.8. *Let $(W, \tilde{g})$ be a connected Riemannian manifold with a boundary $(\partial W, g)$ such that*

(a) $R_{\tilde{g}} \equiv 0$ *on* W,
(b) $\int_{\partial W} h_{\tilde{g}} d\sigma_g > 0$, *where* $g = \tilde{g}|_{\partial W}$.

Then the conformal manifold $(W, [\tilde{g}])$ *cannot be of a zero conformal class.*

Proof. Assume $(W, [\tilde{g}])$ is of a zero conformal class, then we choose a metric $\bar{g} \in [\tilde{g}]$:

$$R_{\bar{g}} \equiv 0 \ \text{ on } W, \quad h_{\bar{g}} \equiv 0 \ \text{ along } \partial W.$$

Then $\tilde{g} = u^{\frac{4}{n-2}} \bar{g}$, and since $R_{\tilde{g}} \equiv 0$, we obtain:

$$\begin{cases} \Delta_{\bar{g}} u & \equiv & 0 & \text{on } W, \\ \partial_\nu u & = & b_n u^\beta h_{\tilde{g}} & \text{on } \partial W, \end{cases}$$

where $b_n = \frac{2(n-1)}{n-2}$, $\beta = \frac{n}{n-2}$. Assume $\int_{\partial W} h_{\tilde{g}} \ d\sigma_g > 0$, then integration by parts gives

$$\int_{\partial W} h_{\tilde{g}} \ d\sigma_g = b_n^{-1} \int_{\partial W} u^{-\beta} \partial_\nu u \ d\sigma_g = -b_n^{-1} \beta \int_W u^{-\beta-1} |\nabla u|^2_{\bar{g}} d\sigma_{\bar{g}} \leq 0.$$

This proves the lemma. $\square$

We need the following straightforward generalization of Lemma 6.8:

Lemma 6.9. *Let* $C > 0$ *be a given constant. Assume that* $(W, \bar{C})$ *is a connected conformal manifold with a boundary* $(\partial W, C)$ *such that for small enough* $\varepsilon > 0$ *there exists a metric* $\tilde{g} \in \bar{C}$ *satisfying the inequalities*

(a) $|R_{\tilde{g}}| < \varepsilon$ *on* W,
(b) $\int_{\partial W} h_{\tilde{g}} d\sigma_g > C > 0$ *along* ∂W, *where* $g = \tilde{g}|_{\partial W}$.

Then the conformal manifold $(W, \bar{C})$ *cannot be of zero conformal class.*

7. Bypass surgery construction

7.1. **The simplest problem to be resolved.** Here we introduce a *bypass surgery contruction* which plays a key role in the proof of Theorem 2.9. Recall that in order to prove Theorem 2.9, we have to find a psc-concordance and a slicing function so that the corresponding Λ-function is non-negative everywhere.

Here we start with resolving the simplest problem of the Λ-function we have to deal with. Let C_0, C_1 be two positive conformal classes on M that are conformally psc-concordant. We choose a conformal psc-concordance $(M \times I, \bar{C})$, a psc-metric $\bar{g} \in \bar{C}$, and a sclicing function $\bar{\alpha} : M \times I \to I$. According to Proposition 3.7, we can alter the conformal psc-concordance $(M \times I, \bar{C})$ by a pseudoisotopy so that $\bar{g} = g_t + dt^2$, and $\bar{\alpha}$ coincides with the projection $\pi_I : M \times I \to I$. We denote by $\Lambda := \Lambda_{(M \times I, \bar{g}, \pi_I)}$ the corresponding Λ-function as above.

As the simplest case, we assume that the function Λ satisfies the following conditions:

$$\begin{cases} \Lambda(0,t) > 0 & \text{if } 0 \leq t < t_0, \ t_1 < t \leq 1, \\ \Lambda(0,t) = 0 & \text{if } t = t_0, t_1, \\ \Lambda(0,t) < 0 & \text{if } t_0 < t < t_0, \end{cases} \tag{21}$$

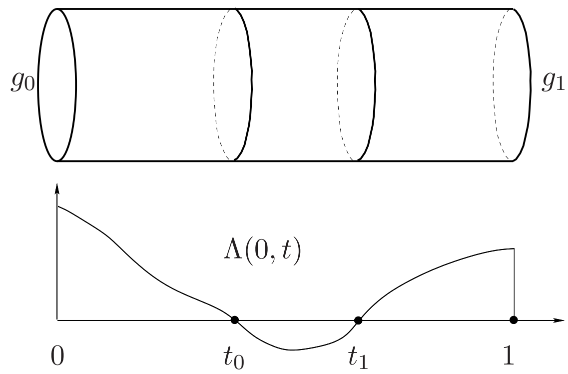

FIGURE 9. The simplest problem to be resolved.

where $t_0 < t_1$ (see Fig. 9). We denote

$$M_\tau = M \times \{\tau\}, \quad \tau = t, s, \quad W^*_{t,s} = M \times [t,s], \quad \bar{g}^*_{t,s} = \bar{g}|_{W^*_{t,s}}.$$

Then we recall that we stretch each cylinder $W^*_{t,s} = M \times [t,s]$ horizontally to get the manifold $(W_{t,s}, \bar{g}_{t,s})$, where $W_{t,s} = M \times [0,1]$.

7.2. **Bypass surgery construction.** Consider the manifolds $(W_{0,t}, \bar{g}_{0,t})$ in more detail. Since the function $\Lambda(0,t)$ satisfies (21), for each t there is a metric metric $\tilde{g}_{0,t} \in \bar{C}_{0,t}$ such that

(1) $R_{\tilde{g}_{0,t}} \equiv 0,\ 0 \le t \le 1$,

(2) $h_{\tilde{g}_{0,t}} = \begin{cases} > 0 & \text{if } 0 < t < t_0, \\ = 0, & \text{if } t = t_0, t_1, \\ < 0 & \text{if } t_0 \le t \le t_1 \end{cases} \qquad \text{along } M_0 \sqcup M_t.$

Here the mean curvature functions $h_{\tilde{g}_{0,t}}$ are smooth for each t, and the function $t \mapsto h_{\tilde{g}_{0,t}}$ is a continiuos function.

We denote by $\tilde{g}$ the restriction of the metric $\tilde{g}_{0,t}$ to $M_0 \sqcup M_t$, and by $d\sigma_{\tilde{g}}$ the corresponding volume form. Then according to Lemma 6.3, we obtain a continuous function

$$\mu_1(0,t) := \mu_1(L_{\bar{g}_{0,t}}) = \int_{M_0 \sqcup M_t} h_{\tilde{g}_{0,t}}\, d\sigma_{\tilde{g}}. \tag{22}$$

By construction, we have the manifold $W_{0,t} = M \times [0,1]$ with the metric $\bar{g}_{0,t}$ which is stretched out of the metric $\bar{g}^*_{0,t} = \bar{g}|_{M\times[0,t]}$. Let us choose a base point $x_0 \in M \times \{0\}$.

We obtain a family of Riemannian manifolds $(M \times [0,1], \bar{g}_{0,t}, x_0)$ which converges to the cylindrical manifold

$$(M \times [0,1], g_0 + dt^2, x_0)$$

as $t \to 0$. In particular, we see that

$$\lim_{t\to 0} \mu_1(L_{\bar{g}_{0,t}}) = \mu_1(L_{g_0+dt^2})$$

is finite, where $\mu_1(L_{g_0+dt^2})$ is a principal eigenvalue of the scalar-flat boundary problem on $(M \times [0,1], g_0 + dt^2)$. Thus the function $\mu_1(0,t)$ is continuous on $[0,1]$.

Let K_0 be the maximum value of the function $t \mapsto |\mu_1(0,t)| + 2,\ 0 \leq t \leq 1$. Now we use Lemma 6.6 to construct the conformal metrics $\tilde{\bar{\hbar}}^{(\ell)} = u^{\frac{4}{n-2}} \bar{\hbar}^{(\ell)}$ on the cylinder $S^{n-1} \times [0,1]$ such that

(i) $R_{\tilde{\hbar}^{(\ell)}} \equiv 0$ on $S^{n-1} \times [0,1]$;

(ii) $\mu_1(L_{\bar{\hbar}^{(\ell)}}) = \displaystyle\int_{S^{n-1}\times\{0,1\}} h_{\tilde{\hbar}^{(\ell)}} d\sigma_{\tilde{\hbar}^{(\ell)}} > 2K_0$.

Then we choose a base point $z_0 \in S^{n-1}$, and a base point $x_0 \in M$ and the intervals

$$I = \{z_0\} \times [0,1] \subset S^{n-1} \times [0,1], \qquad J = \{x_0\} \times [0,1] \subset W_{0,t} = M \times [0,1].$$

We also choose small $\varepsilon > 0$ and an isometry between I and J.

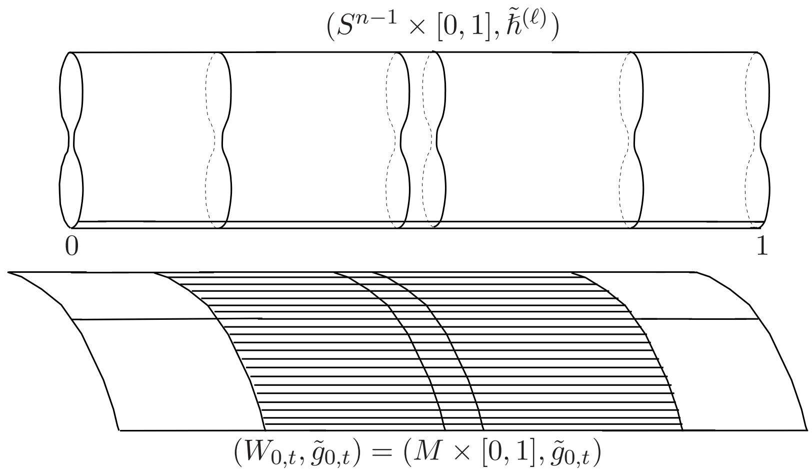

FIGURE 10. Preparation for a bypass surgery.

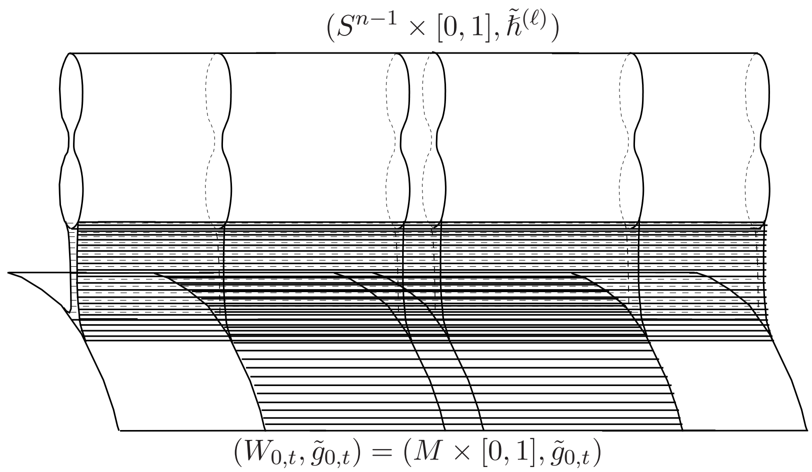

FIGURE 11. The resulting manifold $(\widehat{W}_{0,t}, \widehat{g}_{0,t}(\varepsilon))$ after the bypass surgery.

Now we use Theorem 5.3 to glue together the manifolds

$$(S^{n-1} \times [0,1], \tilde{\hbar}^{(\ell)}) \quad \text{and} \quad (W_{0,t}, \tilde{g}_{0,t})$$

along the intervals $I \cong J$.

We denote the resulting manifold by $(\widehat{W}_{0,t}, \widehat{g}_{0,t}(\varepsilon))$ as it is shown in Fig. 11. We can assume that the metric $\widehat{g}_{0,t}(\varepsilon)$ is such that

(a) $|R_{\widehat{g}_{0,t}(\varepsilon)}| < C'\varepsilon^{n-2}$ on all $\widehat{W}_{0,t}$,

(b) $\displaystyle\int_{\partial\widehat{W}_{0,t}} h_{\widehat{g}_{0,t}}(\varepsilon) d\sigma_{\widehat{g}} > 0$

where C' is some positive constant. Indeed, since (a) holds by Theorem 5.3, for small enough $\varepsilon > 0$, we have

$$\int_{\partial\widehat{W}_{0,t}} h_{\widehat{g}_{0,t}(\varepsilon)} d\sigma_{\widehat{g}} > 2K_0 - K_0 + O(\varepsilon^{n-2}) > 0 \ .$$

Thus the manifold $(\widehat{W}_{0,t}, \widehat{g}_{0,t}(\varepsilon))$ cannot be of zero conformal class for all $t \in (0, 1]$. Since the manifolds $(\widehat{W}_{0,t}, \widehat{g}_{0,t}(\varepsilon))$ are conformally positive for small t, they stay positive for all t by continuity. We summarize the above construction:

Proposition 7.1. *Let $\bar{C}$ be a positive conformal class on $M \times I$, $\bar{g} = g_t + dt^2$ be a metric in $\bar{C}$ minimal along the boundary $M \times \{0, 1\}$, with the projection $\pi_I : M \times I \to I$ playing a role of a slicing function. Assume that at least one of the functions $\Lambda_{0,t}$, $\Lambda_{s,1}$ is negative somewhere. Then there exist constant $\ell > 0$ and small enough $\varepsilon > 0$ such that for the resulting bypass surgery manifold $(\widehat{W}^{(\ell)}_{0,1}, \widehat{g}^{(\ell)}_{0,1}(\varepsilon))$, the manifolds*

$$(\widehat{W}^{(\ell)}_{0,t}, \widehat{g}^{(\ell)}_{0,t}(\varepsilon)) \quad \textit{and} \quad (\widehat{W}^{(\ell)}_{s,1}, \widehat{g}^{(\ell)}_{s,1}(\varepsilon))$$

are positive conformal manifolds for all $0 < t, s < 1$.

Remark. The resulting manifold $(\widehat{W}_{0,1}, \widehat{g}_{0,1}(\varepsilon))$ depends on the constant ℓ which defines the metric $\hbar^{(\ell)}$ and a choice of $\varepsilon > 0$. To emphasize this dependence, we use the notation $(\widehat{W}^{(\ell)}_{0,1}, \widehat{g}^{(\ell)}_{0,1}(\varepsilon))$.

8. Proof of Theorem 2.9

We assume that Theorem 2.9 fails, and we find a countexample (M, C_0, C_1), i.e.

- for any choice of a conformal psc-concordance $(M \times I, \bar{C})$ between postive conformal classes C_0 and C_1,
- for any choice of a metric $\bar{g} \in \bar{C}$ minimal along the boundary $M \times \{0, 1\}$,
- for any choice of a slicing function $\bar{\alpha} : M \times I \to I$

there exist $t < s$, $0 \le t < s \le 1$, such that $\Lambda_{(M\times I,\bar{g},\bar{\alpha})}(t, s) < 0$.

Then we choose psc-metrics $g_0 \in C_0$, $g_1 \in C_1$, and a metric $\bar{g}$ on $M \times I$ with $\lambda_1(L_{\bar{g}}) > 0$. Then, according to Proposition 3.7, we can assume that up to pseudoisotopy, the metric $\bar{g}$ is given as $\bar{g} = g_\tau + d\tau^2$, $0 \le \tau \le 1$.

We denote $W^*_{t,s} = M \times [t, s]$, and $\bar{g}^*_{t,s} = \bar{g}|_{M\times[t,s]}$. Then we use the function

$$\xi_{t,s} : [0, 1] \to [t, s], \quad \tau \mapsto (1 - \tau)t + \tau s$$

to stretch the manifold $(W^*_{t,s}, \bar{g}^*_{t,s})$ to $M \times [0,1]$:

$$\bar{\xi}_{t,s} : M \times I \to M \times [t,s], \quad \bar{\xi}_{t,s}(x,\tau) = (x, \xi_{t,s}(\tau)).$$

We obtain the manifold $(M \times I, \bar{g}_{t.s})$, where $\bar{g}_{t.s} := \bar{\xi}^*_{t,s}(\bar{g}^*_{t,s})$.

Then for each $0 \leq t < s \leq 1$, we consider the conformal Laplacian $L_{\bar{g}_{t,s}}$ with minimal boundary condition, and we obtain the function $\Lambda(t,s) = \lambda_1(L_{\bar{g}_{t,s}})$ as above.

Besides, we consider the conformal Laplacian $L_{\bar{g}_{t,s}}$ with scalar-flatboundary problem and denote by $\mu_1(L_{\bar{g}_{t,s}})$ the principal eigenvalue. We notice that the function

$$(t,s) \mapsto \mu_1(L_{\bar{g}_{t,s}})$$

is continious. Furthemore, since the Riemannian manifolds $(M \times I, \bar{g}_{t,s})$ converge to the cylindrical manifold $(M \times I, g_s + d\tau)$ as $t \to s$, we obtain that the function $\mu_1(L_{\bar{g}_{t,s}})$ is continuous on the closed triangle $\bar{T} = \{ \ (t,s) \mid 0 \leq t \leq s \leq 1 \ \}$. We define

$$K_1 = \max_{(t,s)\in\bar{T}} |\mu_1(L_{\bar{g}_{t,s}})|.$$

Then we find ℓ such that $\mu_1(L_{\bar{\hbar}^{(\ell)}}) > K_1 + 2$, where $\bar{\hbar}^{(\ell)} = \hbar^{(\ell)} + d\tau^2$ is a cylindrical metric on $S^{n-1} \times [0,1]$. Then we make a bypass surgery to get a manifold

$$(\widehat{W}, \widehat{g}) = (M \times [0,1], g_\tau + d\tau^2) \#_{[0,1]} (S^{n-1} \times [0,1], \hbar^{(\ell)} + d\tau^2).$$

We notice that $(\widehat{W}, [\widehat{g}])$ is a conformal psc-concordance between the metrics

$$g_0^{(1)} = g_0 \# \hbar^{(\ell)} \quad \text{and} \quad g_1^{(1)} = g_1 \# \hbar^{(\ell)}.$$

According to Lemma 6.7, the triple $(M, [g_0^{(1)}], [g_1^{(1)}])$ is also a counterexample.

For each $t < s$, we also consider the manifolds

$$(\widehat{W}^*_{t,s}, \widehat{g}^*_{t,s}) = (M \times [t,s, g_\tau + d\tau^2) \#_{[t,s]} (S^{n-1} \times [t,s], \hbar^{(\ell)} + d\tau^2).$$

and then we use the map $\bar{\xi}_{t,s}$ as above to stretch $(\widehat{W}^*_{t,s}, \widehat{g}^*_{t,s})$ to $(\widehat{W}_{t,s}, \widehat{g}_{t,s})$, where

$$(\widehat{W}_{t,s}, \widehat{g}_{t,s}) = (M \times I, \bar{\xi}^*_{t,s}(g_\tau + d\tau^2)) \#_{[0,1]} (S^{n-1} \times [0,1], \bar{\xi}^*_{t,s}(\hbar^{(\ell)} + d\tau^2)).$$

We notice that $\bar{\xi}^*_{t,s}(\hbar^{(\ell)} + d\tau^2) = \hbar^{(\ell)} + d\tau^2$ since the metric $\hbar^{(\ell)} + d\tau^2$ is cylindrical.

We consider the scalar-flat boundary problems on the manifolds

$$(M \times I, \bar{\xi}^*_{t,s}(g_\tau + d\tau^2)) \quad \text{and} \quad (S^{n-1} \times [0,1], \bar{\xi}^*_{t,s}(\hbar^{(\ell)} + d\tau^2))$$

We denote by $\mu_1(L_{g_\tau + d\tau^2)})$ and $\mu_1(L_{\hbar^{(\ell)}})$ corresponding principal eigenvalues. By the choices we made above,

$$\mu_1(L_{g_\tau + d\tau^2)}) + \mu_1(L_{\hbar^{(\ell)}}) > 2$$

Now we consider the scalar-flat boundary problem on the manifold $(\widehat{W}_{t,s}, \widehat{g}_{t,s})$. Since we construct the metric $\widehat{g}_{t,s}$ by taking a connected sum of the metrics $\bar{\xi}^*_{t,s}(g_\tau + d\tau^2))$ and $\hbar^{(\ell)} + d\tau^2$, there is $\varepsilon > 0$, $\varepsilon < 1$, such that

$$|\mu_1(L_{\widehat{g}_{t,s}}) - \mu_1(L_{g_\tau + d\tau^2)}) - \mu_1(L_{\hbar^{(\ell)}})| < \varepsilon$$

Thus $\mu_1(L_{\widehat{g}_{t,s}}) \neq 0$ for all $t < s$. Recall that the function $(t,s) \mapsto \mu_1(L_{\widehat{g}_{t,s}})$ is continuous on the closed traingle $\bar{T} = \{ \ (t,s) \mid 0 \leq t \leq s \leq 1 \ \}$, and $\mu_1(L_{\widehat{g}_{0,1}}) > 0$, thus

$\mu_1(L_{\widehat{g}_{t,s}}) > 0$ for all values $t < s$. Hence $\lambda_1(L_{\widehat{g}_{t,s}}) > 0$ for all t, s, and thus the the triple $(M, [g_0^{(1)}], [g_1^{(1)}])$ is not a counterexample. Contradiction.

9. Surgery Lemma for concordances

The goal of this section is to prove Theorem 2.3. To make the constructions transparent, we describe in detail the case of regular spherical surgery. The almost spherical surgery is very similar.

First, we briefly review the relevant surgery constructions. We follow the scheme given by M. Walsh in great detail, see [31] and also [32].

9.1. **Gromov-Lawson surgery.** Let M be a closed manifold, $\dim M = n-1$, and $S^p \times D^{q+1} \subset M$ be a sphere embedded with a trivial normal bundle, $p+q+1 = n-1$. Let M' be the manifold obtained as the result of surgery on M along S^p:

$$M' = (M \setminus (S^p \times D^{q+1})) \cup_{S^p \times S^q} D^{p+1} \times S^q.$$

We denote $I_0 = [0,1]$. It is convenient to attach the handle $D^{p+1} \times D^{q+1}$ to the cylinder $(M \times I_0)$ to obtain the cobordism V, the trace of surgery between the manifolds M and M':

$$\begin{array}{rcl} V &=& (M \times I_0) \cup_{(S^p \times D^{q+1}) \times \{1\}} D^{p+1} \times D^{q+1}, \\ \\ \partial V &=& M \sqcup -M'. \end{array}$$

Let g be a psc-metric on M. The Gromov-Lawson procedure can be "formalized" as follows. The key step of the Gromov-Lawson construction is to deform the metric g near sphere S^p to the standard metric. This could be done in two standard steps in order to modify the manifold $M \times I_0$, then construct a trace V of this surgery:

(1) We attach the cylinder $S^p \times D^{q+1} \times I_1$ to $M \times I_0$, where we identify

$$S^p \times D^{q+1} \times \{0\} \quad \subset \quad \partial(S^p \times D^{q+1} \times I_1) \quad \text{with}$$

$$S^p \times D^{q+1} \quad \subset \quad M \times \{1\} \subset \partial S^p \times D^{q+1}.$$

According to the Gromov-Lawson deformations, the metric g could be assumed to be already standard near the boundary $S^p \times D^{q+1} \times \{1\}$ of the $S^p \times D^{q+1} \times I_1$, i.e., $h_0 + ds^2$, where h_0 is a round metric on S^p and ds^2 is a flat metric on D^{q+1}.

(2) Next, let D_+^{q+2} be a half of the standard disk in $\mathbf{R}^{q+2}$; in particular, the boundary $\partial D_+^{q+2} = D^{q+1} \cup S_+^{q+1}$. We assume that D_+^{q+2} is equipped with a *standard torpedo metric*, as it is shown at Fig. 12.

Then we attach $S^p \times D_+^{q+2}$ to

$$M \times I_0 \cup S^p \times D^{q+1} \times I_1$$

by identifying

$$S^p \times D^{q+1} \times \{1\} \quad \subset \quad \partial(S^p \times D^{q+1} \times I_1) \quad \text{with}$$

$$S^p \times D^{q+1} \quad \subset \quad \partial(S^p \times D_+^{q+2}), \quad \text{(see Fig. 12).}$$

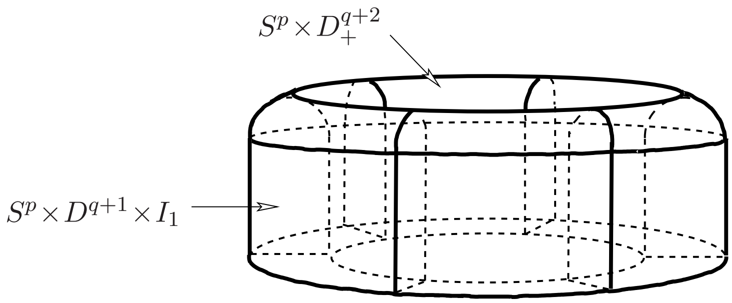

FIGURE 12. The part $(S^p\times D^{q+1}\times I_1)\cup(S^p\times D^{q+2}_+)$ with torpedo metric.

We denote by V_0 the resulting manifold:

$$V_0=(M\times I_0)\cup(S^p\times D^{q+1}\times I_1)\cup(S^p\times D^{q+2}_+),\quad \text{(see Fig. 13, (a))}.$$

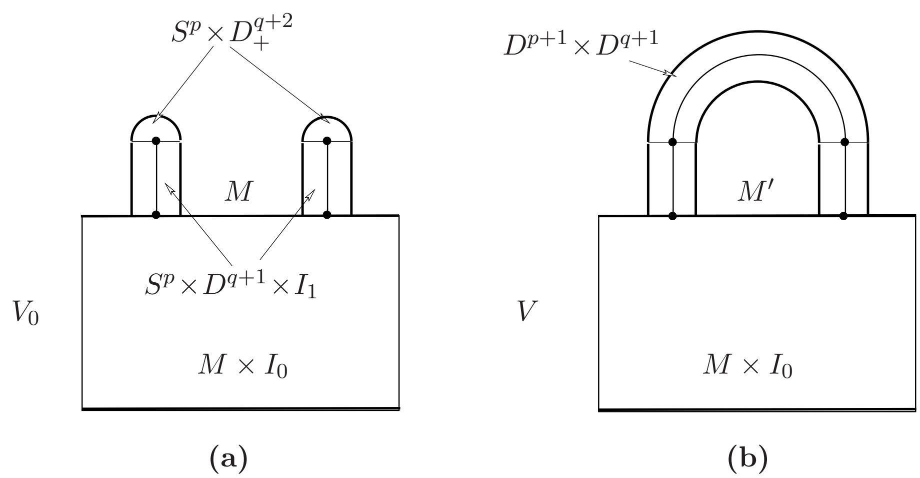

FIGURE 13. Trace of the surgery V between M and M'.

(3) To obtain a trace V of this surgery, we delete the "cup" $S^p\times D^{q+2}_+$ out of V_0 and attach the handle $D^{p+1}\times D^{q+1}$ by identifying

$$S^p\times D^{q+1}\times\{1\}\quad\subset\quad\partial(S^p\times D^{q+1}\times I_1)\quad\text{with}$$

$$S^p\times D^{q+1}\quad\subset\quad\partial(D^{p+1}\times D^{q+1}),\quad\text{(see Fig. 13, (b))}.$$

Thus for a psc-metric g on M, there is a "canonical" psc-metric $\tilde g$ on V, such that $\tilde g$ is a product-metric near the boundary:

$$\begin{aligned}\tilde g &= g+ds^2\quad\text{near } M,\ \text{and}\\ \tilde g &= g'+ds^2\quad\text{near } M',\ \text{with } R_{g'}>0.\end{aligned}\tag{23}$$

Here s is a normal coordinate near the boundary of V.

9.2. **Surgery and psc-isotopy.** This is easy. Let g_t be a smooth family of psc-metrics on M, $t\in[0,1]$. Then the above construction of the metric $\tilde g$ on the manifold V satisfying (23) depends smoothly on the metric g. Thus, we obtain a family of Riemannian

manifolds $(V, \tilde{g}_t)$ such that the restriction $g'_t = \tilde{g}_t|_{M'}$ provides psc-isotopy between g'_0 and g'_1.

9.3. **Surgery and psc-concordance.** Let g_0 and g_1 be two psc-metrics on M. Then we have constructed the Riemannian manifolds $(V, \tilde{g}_0)$ and $(V, \tilde{g}_1)$ as above.

Now we assume that $(M \times [0,1], \bar{g})$ is a psc-concordance between psc-metrics g_0 and g_1. In particular, we assume that we are given $\varepsilon > 0$ such that

$$\bar{g}|_{M\times[0,\varepsilon)} = g_0 + dt^2, \quad \bar{g}|_{M\times(1-\varepsilon,1]} = g_1 + dt^2.$$

First, we would like to extend the psc-concordance $(M \times [0,1], \bar{g})$ to a longer cylinder. We choose $a > 0$ and attach the cylinders

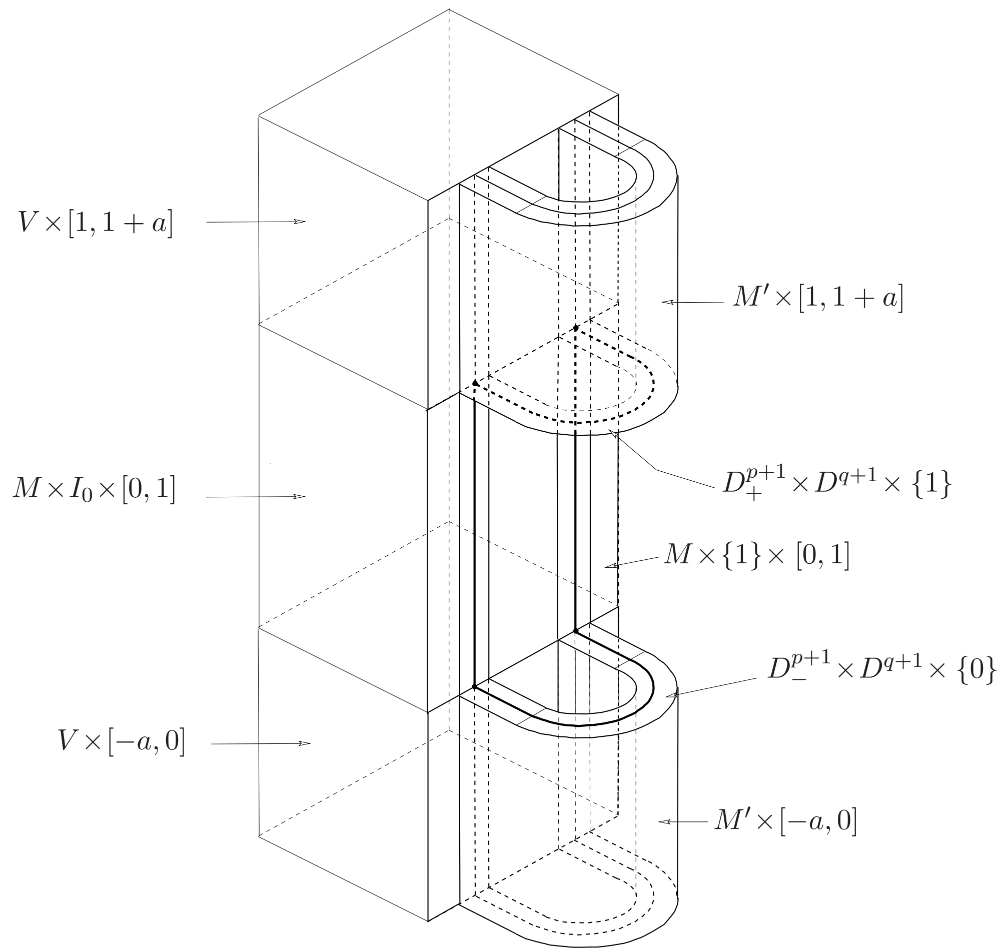

FIGURE 14. The manifold W_0: the first step to construct concordance between g'_0 and g'_1.

$$(M \times [-a,0], g_0 + dt^2) \quad \text{and} \quad (M \times [1,1+a], g_1 + dt^2)$$

to the psc-concordance $(M \times I, \bar{g})$:

$$M \times [-a, 1+a] = (M \times [-a,0]) \cup (M \times [0,1]) \cup (M \times [1,1+a]).$$

We obtain the Riemannian manifold $(M\times[-a,1+a],\hat{g})$, where

$$\hat{g}|_{M\times[-a,0]}=g_0+dt^2,\quad \hat{g}|_{M\times I}=\bar{g},\quad \hat{g}|_{M\times[1,1+a]}=g_1+dt^2.$$

Now we construct the manifold W_0 as follows, see Fig. 14.

$$\begin{array}{rcl} M\times I_0\times\{0\} & \subset & V\times\{0\}\subset V\times[-a,0] \quad \text{with} \\ \\ M\times I_0\times\{0\} & \subset & M\times I_0\times[0,1], \quad \text{and} \\ \\ M\times I_0\times\{1\} & \subset & M\times I_0\times[0,1], \quad \text{with} \\ \\ M\times I_0\times\{1\} & \subset & V\times\{1\}\subset V\times[1,1+a]. \end{array}$$

We glue together the manifolds $V\times[-a,0]$, $(M\times I_0)\times[0,1]$ and $V\times[1,1+a]$ as it is shown at Fig 14, i.e. we identify

$$\begin{array}{rcl} M\times I_0\times\{0\} & \subset & V\times\{0\}\subset V\times[-a,0] \quad \text{with} \\ \\ M\times I_0\times\{0\} & \subset & M\times I_0\times[0,1], \end{array}$$

and also

$$\begin{array}{rcl} M\times I_0\times\{1\} & \subset & M\times I_0\times[0,1], \quad \text{with} \\ \\ M\times I_0\times\{1\} & \subset & V\times\{1\}\subset V\times[1,1+a]. \end{array}$$

We notice that the boundary of the manifold W_0 is decomposed as follows:

$$\partial W_0\cong(V\times\{-a\})\cup(M\times[-a,a+1])\cup(V\times\{1+a\})\cup Y',$$

where the manifold Y' is given as

$$\begin{array}{rcl} Y' & = & (M'\times[-a,0])\cup(D^{p+1}\times D^{q+1}\times\{0\})\cup \\ \\ & & (M\times\{1\}\times[0,1])\cup(D^{p+1}\times D^{q+1}\times\{1\})\cup(M'\times[1,1+a]), \end{array}$$

as it is shown in Fig. 14. We can assume that the psc-concordance $\bar{g}$ on $M\times I$ and its extension, the metric $\hat{g}$ to $M\times[-a,1+a]$, are standard on the strip

$$S^p\times D^{q+1}\times[-a,1+a].$$

Thus we obtain a psc-metric G_0 on W_0, and can assume that G_0 is a product metric on the submanifold $M\times I_0\times[-a,1+a]$ and standard on the handles

$$(D^{p+1}\times D^{q+1})\times[-a,0]\ \text{ and }\ (D^{p+1}\times D^{q+1})\times[1,1+a]).$$

We would like to perform the second surgery, this time on the manifold Y', in such a way that the resulting manifold will be diffeomorphic to the cylinder $M'\times[-a,1+a]$.

We notice that we are given a canonical embedding $S^{p+1}\times D^{q+1}\subset Y'$. Here, the sphere S^{p+1} is decomposed as follows:

$$S^{p+1}=(D^{p+1}_{-}\times\{0\})\cup(S^p\times[0,1])\cup(D^{p+1}_{+}\times\{1\})$$

(see Fig. 15). Clearly the induced metric h on S^{p+1} is not standard; however, after smoothing corners, the metric h on S^{p+1} is given by stretching and “bending twice” the standard metric (see Fig. 15). Next, in order to turn the metric on $S^{p+1}\times D^{q+1}$ into

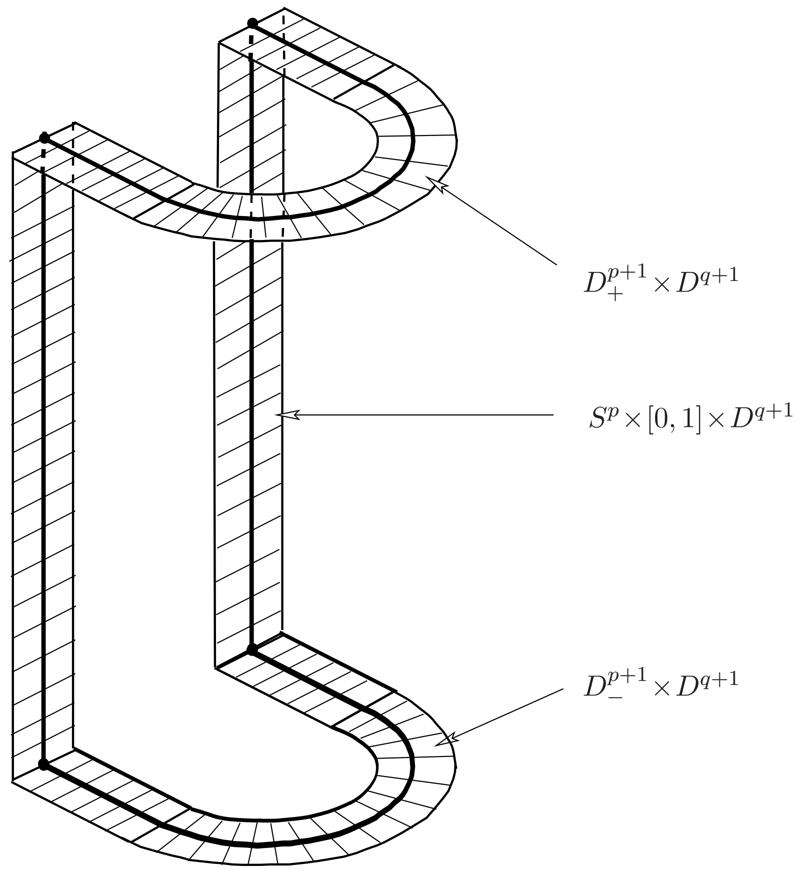

FIGURE 15. The embedding $S^{p+1} \times D^{q+1}$ to Y'

standard, torpedo metric, we attach the cylinder $S^{p+1} \times D^{q+1} \times I_1$ and after that the handle $S^{p+1} \times D_+^{q+2}$ with the "topedo" metric as it is shown in Fig 16. Here $\partial D_+^{q+2} = D^{q+1} \cup S_+^{q+1}$, where S_+^{q+1} is a hemisphere equipped with a torpedo metric. We identify:

$$S^{p+1} \times D^{q+1} \times \{0\} \subset S^{p+1} \times D^{q+1} \times I_1 \text{ with}$$

$$S^{p+1} \times D^{q+1} \subset Y; \text{ and then}$$

$$S^{p+1} \times D^{q+1} \subset S^{p+1} \times (D^{q+1} \cup S_+^{q+1}) = \partial D_+^{q+2} \text{ with}$$

$$S^{p+1} \times D^{q+1} \times \{1\} \subset S^{p+1} \times D^{q+1} \times I_1,$$

(see Fig. 16). The resulting manifold W_1 is "surgery-ready".

To perform the surgery, we just delete the manifold $S^{p+1} \times D_+^{q+2}$ and instead attach the handle $D^{p+2} \times D^{q+1}$ to W_1 by identifying the manifolds

$$S^{p+1} \times D^{q+1} \times \{1\} \subset S^{p+1} \times D^{q+1} \times I_1 \quad \text{with}$$

$$S^{p+1} \times D^{q+1} \subset \partial(D^{p+2} \times D^{q+1}) \subset D^{p+2} \times D^{q+1}.$$

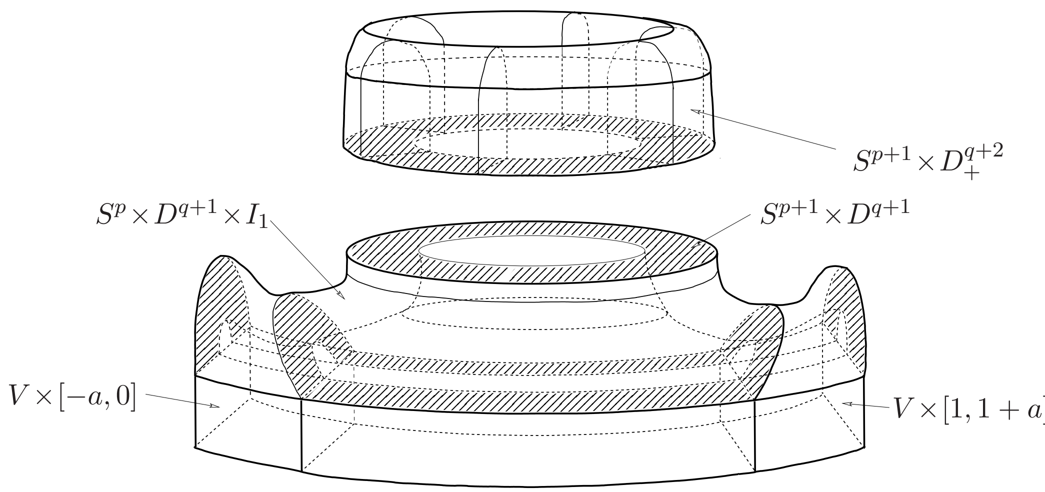

FIGURE 16. Preparation for the second surgery

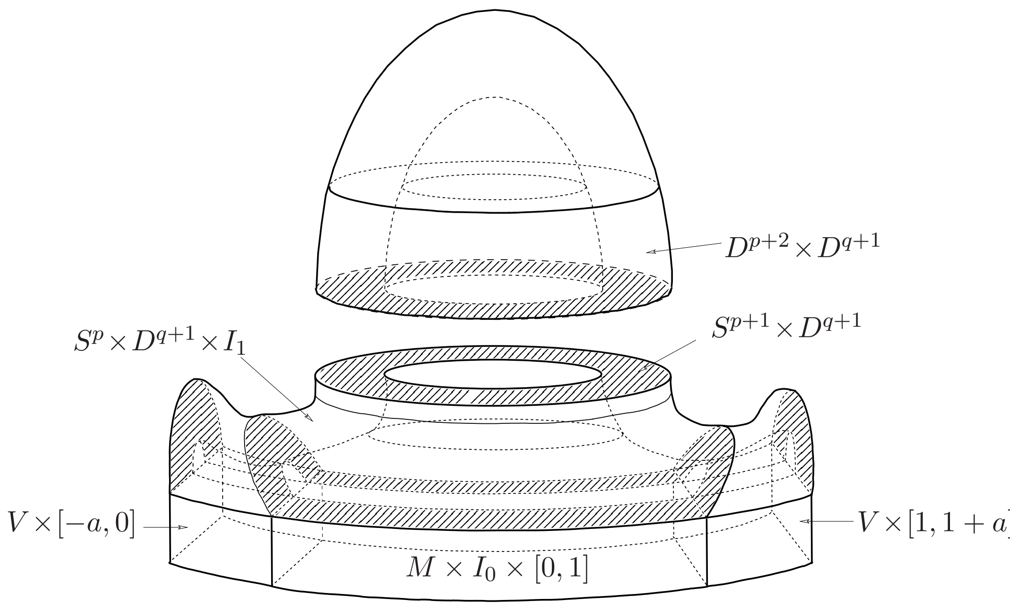

FIGURE 17. The second surgery

We denote by W the resulting manifold: $W = W_1 \cup D^{p+2} \times D^{q+1}$ (see Fig 17). One can easily see that W is diffeomorphic to $V \times [-a, 1+a]$, in particular,

$$\partial W \cong (V \times \{-a\}) \cup (M \times [-a, 1+a]) \cup (V \times \{1+a\}) \cup (M' \times [-a, 1+a]),$$

and the manifold $M' \times [-a, 1+a]$ is given a psc-metric $\bar{g}'$, so that $(M' \times [-a, 1+a], \bar{g}')$ is a psc-concordance joining the psc-metrics g_0' and g_1' on M'. This shows that if g_0, g_1 are psc-concordant psc-metrics on M, then the Gromov-Lawson construction yields psc-concordant psc-metrics g_0', g_1' on M'. This completes the proof of Theorem 2.3.

Appendix

Here we prove Lemma 6.6:

Lemma 6.6. *Let* $(S^{n-1}\times I, \bar{h}^{(\ell)})$*, where* $\bar{h}^{(\ell)} = h^{(\ell)} + dt^2$ *be the cylinder. Let* $C>0$ *be a given constant. Then for any* ℓ *satisfying* $\frac{3}{4}a_n\lambda_1(\mathcal{L}_{h^{(\ell)}}) > C^2$*, there exists a metric* $\tilde{h}^{(\ell)} \in [\bar{h}^{(\ell)}]$ *such that*

(1) $R_{\tilde{h}^{(\ell)}} \equiv 0$,
(2) $\mu_1(L_{h^{(\ell)}}) = \int_{S^{n-1}\times\{0,1\}} h_{\tilde{h}^{(\ell)}} d\sigma_{\tilde{h}^{(\ell)}} \geq C$.

Proof. Since $\bar{h}^{(\ell)}$ is a product metric, the mean curvature vanishes. Hence for the cylinder $(S^{n-1}\times I, \bar{h}^{(\ell)})$, the energy functional is given as

$$E_{\bar{h}^{(\ell)}}(v) = \int_0^1 \int_{S^{n-1}} \left(a_n|\nabla_{\bar{h}^{(\ell)}} v|^2 + R_{\bar{h}^{(\ell)}} v^2\right) d\sigma_{h^{(\ell)}} dt.$$

The coresponding principal eigenvalue is given then as

$$\mu_1 = \inf\{\ E_{\bar{h}^{(\ell)}}(v) \mid \|v\|^2_{L_2(\partial)} = 1\ \}, \quad \text{where} \quad \|v\|^2_{L_2(\partial)} = \int_{S^{n-1}\times\{0,1\}} v^2 d\sigma_{h^{(\ell)}}.$$

Let $v>0$ be an eigenfunction corresponding to the eigenvalue μ_1 as above, i.e.,

$$\mu_1 = E_{\bar{h}^{(\ell)}}(v), \quad \|v\|^2_{L_2(\partial)} = \int_{S^{n-1}} \left(v^2(x,1)+v^2(x,0)\right) d\sigma_{h^{(\ell)}} = 1, \tag{24}$$

We denote by $v' = v'(x,t)$ the partial derivative with respect to t. Then we have:

$$\begin{aligned} E_{\bar{h}^{(\ell)}}(v) &= \int_0^1 \int_{S^{n-1}} \left(a_n|\nabla_{\bar{h}^{(\ell)}} v|^2 + R_{\bar{h}^{(\ell)}} v^2\right) d\sigma_{h^{(\ell)}} dt \\ &= \int_0^1 \int_{S^{n-1}} \left(\left(a_n|\nabla_{h^{(\ell)}} v|^2 + R_{h^{(\ell)}} v^2\right) + a_n(v')^2\right) d\sigma_{h^{(\ell)}} dt. \end{aligned} \tag{25}$$

Let $u_0>0$ be an eigenfunction on S^{n-1} corresponding to $\lambda_1 = \lambda_1(\mathcal{L}_{h^{(\ell)}})$, i.e.,

$$\mathcal{L}_{h^{(\ell)}} u_0 = \lambda_1 u_0, \quad \int_{S^{n-1}} u_0^2 \, d\sigma_{h^{(\ell)}} = 1. \tag{26}$$

Since u_0 is normalized as in (26), we obtain the inequality

$$\frac{\int_{S^{n-1}} (\mathcal{L}_{h^{(\ell)}} v)\, v \ d\sigma_{h^{(\ell)}}}{\int_{S^{n-1}} v^2 d\sigma_{h^{(\ell)}}} \geq \frac{\int_{S^{n-1}} (\mathcal{L}_{h^{(\ell)}} u_0)\, u_0 \ d\sigma_{h^{(\ell)}}}{\int_{S^{n-1}} u_0^2 d\sigma_{h^{(\ell)}}} = \lambda_1 \ ,$$

where we regard v as a function of x under given t. Then we continue (25):

$$
\begin{array}{rcl}
E_{\bar{\hbar}(\ell)}(v) & = & \displaystyle\int_0^1 \int_{S^{n-1}} \left(\left(a_n|\nabla_{\hbar(\ell)} v|^2 + R_{\hbar(\ell)} v^2\right) + a_n(v')^2 \right) d\sigma_{\hbar(\ell)} dt \\
& = & \displaystyle\int_0^1 \int_{S^{n-1}} \left((\mathcal{L}_{\hbar(\ell)} v)\, v + a_n(v')^2 \right) d\sigma_{\hbar(\ell)} dt \\
& \geq & \displaystyle\int_0^1 \int_{S^{n-1}} \left(\lambda_1 v^2 + a_n(v')^2 \right) d\sigma_{\hbar(\ell)} dt \\
& = & \displaystyle\int_{S^{n-1}} \left(\int_0^1 \left(\lambda_1 v^2 + a_n(v')^2 \right) dt \right) d\sigma_{\hbar(\ell)} \ ,
\end{array} \tag{27}
$$

where at the end we have changed the order of integration.

Now we consider the following one-dimensional elementary problem. Namely, we would like to find a minimal value of the functional

$$
\mathcal{E}(f) = \int_0^1 ((f')^2 + k^2 f^2) dt, \quad \text{provided} \quad f(0) = a, \quad f(1) = b. \tag{28}
$$

It is well-known that the functional (28) gives the Euler-Lagrange equation:

$$
f'' - k^2 f = 0, \quad \text{provided} \quad f(0) = a, \quad f(1) = b. \tag{29}
$$

Moreover, a solution f_0 of (29) minimizes the functional (28). To find the solution f_0, we write a general solution as $f = Ae^{kt} + B^{-kt}$. Then the boundary conditions give:

$$
\left\{ \begin{array}{lcl} A + B & = & a \\ Ae^k + Be^{-k} & = & b \end{array} \right. \Longrightarrow \left\{ \begin{array}{lcl} A + B & = & a \\ Ae^{2k} + B & = & be^k \end{array} \right. \Longrightarrow \left\{ \begin{array}{lcl} A & = & \frac{be^k - a}{e^{2k}-1} \\ B & = & \frac{ae^{2k} - be^k}{e^{2k}-1} \end{array} \right. \tag{30}
$$

The solution f_0 is then given as

$$
f_0 = Ae^{kt} + Be^{-kt} = \frac{be^k - a}{e^{2k} - 1} e^{kt} + \frac{ae^{2k} - be^k}{e^{2k} - 1} e^{-kt}.
$$

Then we compute $\mathcal{E}(f_0)$:

$$
\begin{array}{rcl}
\mathcal{E}(f_0) & = & \int_0^1 ((f_0')^2 + k^2 f_0^2) dt, \\
& = & \int_0^1 \left(k^2(Ae^{kt} - Be^{-kt})^2 + k^2(Ae^{kt} + Be^{-kt})^2 \right) dt \\
& = & 2k^2 \int_0^1 \left(A^2 e^{2kt} + B^2 e^{-2kt} \right) dt \\
& = & k\left[(A^2(e^{2k} - 1) + e^{-2k} B^2 (e^{2k} - 1) \right]
\end{array}
$$

We substitute A and B from (30) to continue:

$$\begin{aligned}
\mathcal{E}(f_0) &= k\left[(A^2(e^{2k}-1)+e^{-2k}B^2(e^{2k}-1)\right] \\
&= k\left[\tfrac{(be^k-a)^2}{e^{2k}-1}+\tfrac{e^{-2k}(ae^{2k}-be^k)^2}{e^{2k}-1}\right] \\
&= \tfrac{k}{e^{2k}-1}\left[b^2(e^{2k}+1)-4abe^k+a^2(e^{2k}+1)\right] \\
&= \tfrac{k(e^{2k}+1)}{e^{2k}-1}\left[a^2+b^2-4ab\tfrac{e^k}{e^{2k}+1}\right]
\end{aligned}$$

We denote $\xi=\frac{e^k}{e^{2k}+1}$, then $-4ab\xi\geq -2\xi(a^2+b^2)$. According to (20), $\xi<1/4$, then

$$\begin{aligned}
\mathcal{E}(f_0) &\geq \tfrac{k(e^{2k}+1)}{e^{2k}-1}(a^2+b^2-4ab\xi) \\
&\geq \tfrac{3k}{4}(a^2+b^2).
\end{aligned} \tag{31}$$

We go back to the estimate (27). Let $k^2=\frac{\lambda_1}{a_n}$. For each point $x\in S^{n-1}$, we use the function $f(t)=v(x,t)$. Then we use (31) to see:

$$\begin{aligned}
E_{\bar{\hbar}^{(\ell)}}(v) &\geq \int_{S^{n-1}}\left(\int_0^1(\lambda_1 v^2+a_n(v')^2)\,dt\right)d\sigma_{\hbar^{(\ell)}} \\
&= a_n\int_{S^{n-1}}\left(\int_0^1((v'(x,t))^2+k^2v^2(x,t))\,dt\right)d\sigma_{\hbar^{(\ell)}} \\
&\geq \tfrac{3}{4}a_n\int_{S^{n-1}\times\{0,1\}}k(v^2(x,1)+v^2(x,0))d\sigma_{\hbar^{(\ell)}}=\tfrac{3}{4}\sqrt{a_n\lambda_1}
\end{aligned} \tag{32}$$

since $\lambda_1=a_nk^2$ and $\|v\|^2_{L_2(\partial)}=1$. This completes the proof. $\square$

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